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Entropy of random coverings and 4D quantum gravity

C. Bartocci [a], U. Bruzzo [b], M. Carfora [b,c,*], A. Marzuoli [d,c]

[a] *Dipartimento di Matematica Universitá di Genova, Via L.B. Alberti, 4 16132 Genova, Italy*

[b] *International School for Advanced Studies, SISSA-ISAS, Via Beirut 2-4, 34013 Trieste, Italy*

[c] *Istituto Nazionale di Fisica Nucleare, Sezione di Pavia, Italy*

[d] *Dipartimento di Fisica Nucleare e Teorica dell'Universitàdi Pavia, Via Bassi 6, I-27100 Pavia, Italy*

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Abstract

We discuss the counting of minimal geodesic ball coverings of n-dimensional ($n \geq 3$) riemannian manifolds of bounded geometry, fixed Euler characteristic, and Reidemeister torsion in a given representation of the fundamental group. This counting bears relevance to the analysis of the continuum limit of discrete models of quantum gravity. We establish the conditions under which the number of coverings grows exponentially with the volume, thus allowing for the search of a continuum limit of the corresponding discretized models. The resulting entropy estimates depend on representations of the fundamental group of the manifold through the corresponding Reidemeister torsion. We discuss the sum over inequivalent representations both in the two-dimensional and in the four-dimensional case. Explicit entropy functions as well as significant bounds on the associated critical exponents are obtained in both cases.

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* Corresponding author. E-mail: carfora@pavia.infn.it.

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1. Introduction

Dynamical triangulations, [ADF,D2,Ka,We], have recently attracted much interest as a computationally manageable method for the investigation of discrete models of quantum gravity. This approach deals with a variant of Regge calculus [R,Wi] where, in alternative to the standard usage, the edge lengths of the triangulated manifolds are kept fixed and set equal to some minimal short-distance cut-off, whereas the underlying combinatorial structure of the triangulations takes the role of a statistical variable, varying in some ensemble of manifolds contributing to the model. A dynamical content is thus given to the connectivity of the triangulation in such a way that each choice of a triangulation corresponds to a choice of metric by Regge calculus.

This particular prominence given to the enumeration of triangulations gives to dynamically triangulated gravity the seemingly simple flavor of a combinatorial theory. However, it must be stressed that this simplicity is largely apparent rather than actual, since at a classical level and at a variance with standard Regge calculus, dynamical triangulations do not afford a simple procedure for recovering the Einstein–Hilbert action out of its combinatorial counterpart, diffeomorphism invariance being now completely lost.

The possible advantages in the use of dynamical triangulations are rather related to the different way in which one realizes, in this approach, the sampling of inequivalent riemannian structures. This is obtained by choosing a representative metric (by fixing the edge lengths) and by ergodically varying the combinatorial structure of the triangulation. We do not know of a proof which explicitly shows a correspondence between this procedure and a suitable continuous way of parametrizing the set of inequivalent riemannian structures. Perhaps the Gromov–Hausdorff topology discussed below provides such a correspondence. In any case, it is more or less tacitly assumed that in this way one sweeps a much larger set of riemannian structures as compared to the Regge case, where the formalism, in this respect, is less flexible owing to the constraints expressed by the triangular inequalities. These constraints tend to localize the edge length varying triangulations used in Regge calculus in a neighborhood of the riemannian structure corresponding to the triangulation originally given. Whereas, one expects that the set of discretized manifolds considered in the dynamically triangulated approach is uniformly distributed over the space of all riemannian structures.

This is very appealing for discussing the phase structure in the space of the coupling constants of the theory: the cosmological constant, and the gravitational coupling constant. By defining the regularized partition function as a sum over topologically equivalent triangulations, results for continuum quantum gravity can be extracted by looking for critical points, in the space of coupling constants, where the observables of the model, such as the average number of simplices, diverge and obey scaling relations. This scaling behavior allows for a renormalization of the couplings in terms of the given edge length of the simplices so as to obtain finite values for the volume and other simple geometrical quantities characterizing the extended configurations dominating the theory in the continuum limit. In other words, one looks for the onset of a regime where the details of the simplicial approximation become irrelevant and a continuum theory can be constructed.

There is a general comment that should be made at this stage. In order to provide general entropy estimates for discretized manifolds, we find expedient to introduce another kind of discretization, yet, besides dynamical triangulations and Regge calculus. This discretization is associated with metric ball coverings of given radius. While not so useful from a numerical point of view, it provides a good analytical edge on discrete quantum gravity. It blends the simple combinatorial structure of dynamical triangulations with the deep geometrical content of Regge calculus. We feel that such variety of possible models should be considered with a positive attitude, by taking advantage of the respective good properties rather than emphasizing the drawbacks, as is often done. Thus, even if in what follows we emphasize dynamical triangulations versus Regge calculus, this does not mean that we wish to privilege that formalism with respect to the other. The issue we address, the counting of the number of topologically equivalent discretizations of an n-manifold of given volume ($n \geq 3$) is present in both cases (see [Fro]), but it has been recently mostly emphasized for dynamical triangulations.

As is well known, the main development of discrete models of quantum gravity, and in particular of dynamically triangulated gravity, has resulted from their role in providing a method for regularizing non-critical bosonic string theory (see e.g., [FRS] for a review). This latter can be seen as two-dimensional quantum gravity interacting with D scalar fields, where D is the dimension of the space where the string is embedded. The associated dynamically triangulated models correctly reproduce, in the continuum limit, the results obtained by conformal field theory. In particular, they are consistent with the computation [KPZ], in the context of the Liouville model, of the entropy of closed surfaces with Euler characteristic χ, area A and interacting with matter fields with central charge $c \leq 1$, viz.,

$$S_{\chi}(A) \simeq (\Lambda)^{A} A^{(\chi(\Sigma)/2)(\gamma_{\mathrm{str}}-2)-1}, \tag{1}$$

where Λ is a suitable constant and γ_{str}, the *string exponent*, is given as a function of the central charge by

$$\gamma_{\mathrm{str}} = \tfrac{1}{12}\left(c - 1 - \sqrt{(25-c)(1-c)}\right). \tag{2}$$

The above expression for γ_{str} is valid as long as $c \leq 1$, and it appears to make sense only in the weak coupling phase corresponding to c (or equivalently D) smaller than 1. For $c > 1$, conformal field theory becomes unstable, and the above expression for the string exponent is no longer reliable (recently, an extension of the KZP scaling to the $c > 1$ case has been proposed by Martellini et al. [MSY]). Roughly speaking, it is believed that in this regime the surfaces develop spikes and long tubes, and as seen from a large distance the surface is no longer a two-dimensional object. It collapses into a branched polymers configuration [DFJ]. It is important to stress that two-dimensional dynamically triangulated models are well defined also in these cases, where conformal field theory is no longer trustworthy, and they provide a technique accessible to computer simulations.

A natural question concerns the possibility of extending the techniques and some of the general results of the two-dimensional case to the dimension three and four. This research program has been undertaken by various groups by performing extensive computer simulations of three- and four-dimensional triangulated manifolds.

Although these simulated systems have a rather small size as compared to the simulations used for 2D-gravity (typically one puts together 10^4 four-simplices, whereas in the two-dimensional case triangulations with 10^7 triangles are not unusual [Ag]) interesting results about critical phenomena already emerge (see [D1] for an excellent review). Such results are qualitatively similar in the 3D and 4D cases [Ag,AJ,Va] in the sense that the phase diagram of the theory as a function of the cosmological constant and the gravitational coupling constant shows the existence of a critical point. Here, the configurations dominating the statistical sum change from being crumpled non-extended objects to extended, finite Hausdorff-dimensional, objects. In three dimensions there is a rather strong evidence that this change is associated with a first-order transition indicating the absence of a continuum limit. Whereas, in four dimensions computer simulations indicate that the transition between the crumpled and the extended phases may be of a continuous nature.

There is increasing evidence to the soundness of this picture, and at least from a general foundational point of view, dynamically triangulated gravity seems to be now well established also in dimension three and four. However, there still remain some outstanding problems. The most obvious one is to obtain explicit analytic control on the theory (here we do not consider as dynamically triangulated models the formulations of 3D-gravity à la Ponzano–Regge). It is not yet known if it is possible to obtain such a control, and the best results at the moment come from an interplay between computer simulations and the general analytic properties of the various models considered (e.g., the choice of the most appropriate measure on the set of triangulated manifolds [BM]).

The experience with the two-dimensional case shows that the delicate point here is to ascertain if the number of dynamically triangulated n-manifolds ($n > 2$) of given volume and fixed topology grows with the volume at most at an exponential rate. This is a basic entropy bound necessary for having the correct convergence properties of the partition function defining the model.

In the case of surfaces, the required entropy bounds, such as (1), are provided either by direct counting arguments, or by quantum field theory techniques [BIZ,FRS] as applied to graph enumeration, a technique that has found utility in a number of far reaching applications in surface theory [Wt1,Ko,Pe]. In higher dimensions, the natural generalizations of such approaches are not viable even if numerical as well as some analytical evidence

[Am,ADF,Ag] shows that exponential bounds do hold in simple situations (typically for manifolds with n-sphere topology). Recently, it has even been argued, on the basis of some numerical evidence, that an exponential bound may fail to hold in dimension four [CKR], but this analysis is quite controversial [AJ]. Conversely, Boulatov has provided a nice argument for proving that for a dynamically triangulated homotopy three-sphere there is an exponential bound [Bou] (the constants in the estimates are not characterized, however). Thus, a systematic method for providing explicit entropic bounds relating topology to the number of topologically equivalent triangulations appears as a major open issue in higher dimensional dynamically triangulated gravity [D1].

Without any control on the topology of manifolds, there is no hope in the search for an exponentially bounded entropy function for the number of equivalent triangulations. For instance, it can be shown [Am] that the number of distinct triangulations on (three)-manifolds, with given volume V and arbitrary topological type, grows at least factorially with V. Thus suitable constraints on the class of riemannian manifolds considered are necessary for having exponential growth of the number of equivalent triangulations.

By analogy with the two-dimensional case, one may simply fix the topology a priori (e.g., an n-sphere topology, $n = 3$, $n = 4$). This is a pragmatic point of view. It has the advantage of simplicity, but it has the serious drawback that it does not allow to easily deal with fluctuating topologies, either because it is difficult to know a priori what kind of topological invariants are going to enter the entropy estimates in dimension $n \geq 3$, or because a topological classification of the relevant class of manifolds is often lacking, e.g., in the case of three-manifolds.

The point of view implicit in the approach above is also motivated by the assumption that the topology of a manifold is not apparently under control in terms of the geometrical invariants characterizing the size of a manifold (and hence its entropy) namely the volume or other simple geometrical elements such as the diameter, and bounds on curvatures.

However, the experience with recent developments in riemannian geometry may suggest a change of this restrictive viewpoint. Such an indication comes from a basic theorem due to Cheeger (see e.g., [Ch] for a readable account of such finiteness theorems) according to which, for any given dimension, there are a *finite number of homeomorphism types* in the set of compact riemannian manifolds with volume bounded below, diameter bounded above and sectional curvature bounded in absolute value. Further finiteness results of this type, even under weaker control on the size of the manifolds, have been obtained [Pt,GPW], recently. A typical example in this sense is afforded by considering for arbitrary $r \in \mathbb{R}$, $v \in \mathbb{R}^+$, $D \in \mathbb{R}^+$ and integers $n \geq 3$, the set of closed connected riemannian n-manifolds, M whose sectional curvatures satisfy $\sec(M) \geq r$, whose volume satisfies $\mathrm{Vol}(M) \geq v$, and whose diameter is bounded above by D, $\mathrm{diam}(M) \leq D$. This is an infinite-dimensional collection of riemannian structures, with different underlying topologies. A huge space, for which one can prove finiteness of the homotopy types (in any dimension), finiteness of the homeomorphism types (in dimension $n = 4$), and finiteness of diffeomorphism types (in any dimension $n \geq 5$).

Even more generally, one may consider the set of all metric spaces (smooth manifolds, and more general spaces, e.g., negatively curved polyhedra) of Hausdorff dimension bounded above and for which a (Toponogov's) comparison theorem for geodesic triangles locally holds (Aleksandrov spaces with curvature bounded below [BGP,Per]). On a strict geometrical side, we wish to stress that these are the spaces which arise naturally if one wishes to consider simplicial approximations to riemannian manifolds.

It must be stressed that the imposition of (lower) bounds on sectional curvatures does not seem to be fully consistent with the generic triangulations considered in dynamically triangulated models of quantum gravity. A simple two-dimensional example is afforded by noticing that the local contribution to curvature corresponding to a given vertex is $\pi/3(6-d)$, where d is the order of the vertex (i.e., the number of edges meeting at it). A priori, when considering dynamical triangulations, there is no natural bound to the order d, and the local curvature may grow arbitrarily large. Thus spaces of bounded geometry may appear quite unsuitable as an arena for discussing dynamically triangulated models.

The fact is that the use of spaces of bounded geometry should be considered simply as a technical step needed in order to get definite mathematical control on problems raised when dealing with enumerative problems for dynamical triangulations. In particular, once the entropy estimates are obtained, we should remove the dependence on the cut-offs artificially introduced. A priori, this removal would call for a rather delicate (inductive) limiting procedure, viz., considering the behavior of the sequence of entropy estimates on the nested collection of spaces of bounded geometry obtained by letting the lower bound to the curvature go to (minus) infinity. Actually, the entropy bounds obtained by us turn out to be not sensible to the cut-offs, and the potential shortcomings of the use of spaces of bounded geometry do not appear.

The possibility of getting some mathematical control on the entropy problem by using spaces of bounded geometry is suggested by the topological finiteness results recalled above. To clarify somehow this assertion, let us recall that in any given dimension the set of manifolds which satisfies the hypothesis of these finiteness theorems has a compact closure in a Hausdorff-like topology [Gr1]. This topology is naturally adapted to the coarse grained point of view implicit in the discrete approaches to quantum gravity, thus one may reasonably assume that partition functions associated with such discrete models are continuous in such topology. Since the configuration space is compact, and the partition functions are continuous, it follows that out of the sequence of bounded partition functions corresponding to finer and finer triangulations, we can extract a converging subsequence. This implies the corresponding existence of well-behaved entropy bounds.

Obviously, this is a heuristic argument, which however may serve as a guiding principle. Indeed, following this viewpoint, we proved [CM4] that, *up to a sum over inequivalent orthogonal representations of the fundamental group*, it is possible to explicitly provide the entropy function counting the topologically equivalent ways of covering and packing, with metric balls of given radius, n-manifolds of bounded geometry, for any $n \geq 3$ (notice that here topological equivalence stands for simple-homotopy equivalence). Strictly speaking this is not the entropy function for dynamical triangulations of the given manifold. However, it is easily seen (see the following paragraph) that with a dynamically triangulated manifold there is naturally associated a metric ball covering, and that the number of topologically equivalent metric ball coverings of given radius is not-smaller than the number of corresponding dynamical triangulations. Thus, the entropy function determined in [CM4] is an upper bound to the entropy function for dynamical triangulations (for manifolds of bounded geometry). This argument is useful for establishing that one has exponential bounds on the number of equivalent triangulations. However, it is important to stress that *it does not allow to determine the critical exponents for dynamically triangulated models (for $n \geq 3$)*. As a matter of fact, already for $n = 2$, critical exponents for geodesic ball coverings can be quite different from the γ_{str} appearing in (1). This may be seen as a rather obvious consequence of the intuitive fact that there are many more states accessible to coverings rather than to triangulations, since the latter are combinatorially more rigid.

The analysis in [CM4] was rather incomplete, in particular we did not attempt any explicit determination of the critical exponents for geodesic ball coverings, and the connection between this type of discretization and the more familiar ones, like Regge calculus and dynamical triangulations, was quite unclear. Here we carry out an important step in this direction by explicitly providing an entropy estimate for geodesic ball coverings of four-dimensional manifolds and by determining bounds to the corresponding critical exponent. On passing, we also discuss the two-dimensional case, again by explicitly determining entropy estimates and bounds for the critical exponents.

1.1. Summary of the results

The results obtained can be summarized as follows.

Entropy estimates in a given representation of the fundamental group. Let M be an n-dimensional manifold ($n \geq 2$) of given fundamental group $\pi_1(M)$, and let $[\theta] \in \mathrm{Hom}(\pi_1(M), G)/G$ denote a conjugacy class of representations of $\pi_1(M)$ into a Lie group endowed with an Ad-invariant, symmetric, non-degenerate bilinear form (i.e., with an Ad-invariant metric).

We think of M as generated by a configuration of λ metric balls $\{B(i)\}$ of fixed radius ϵ in such a way that the ϵ-balls cover M while the $\frac{1}{2}\epsilon$-balls are disjoint. Moreover, at most d balls are allowed to mutually overlap, (such d depends on the geometry of the underlying manifold, but it is otherwise independent from ϵ). We refer to the set of balls with radius $\frac{1}{2}\epsilon$ as an $\frac{1}{2}\epsilon$-*geodesic ball packing* of M, while the same set of balls with radius ϵ defines the corresponding ϵ-*geodesic ball covering* of M.

A priori, the balls are topologically non-trivial, namely both the balls themselves and their mutual intersections are not assumed to be contractible (this allows for arbitrarily large positive curvature in the underlying manifold). Explicitly, the non-trivial topology of the balls is described by their twisted cohomology groups $H^*_{\mathfrak{g}}$ with coefficients in a certain (adjoint) flat bundle associated with the representation θ. Roughly speaking, such groups provide *colors* to the balls of the covering, and it is assumed that there are λ inequivalent colors to distribute over the λ balls. Any two such colorings are considered combinatorially inequivalent if the resulting patterns of the balls belong to distinct orbits of the action of the symmetric group acting on the (centers of the) balls. We prove that such combinatorially inequivalent colorings can be used to construct, in the given representation $\theta : \pi_1(M) \to G$, the *distinct minimal geodesic ball coverings* of M, and thus, according to the previous remarks, they can also be used to enumerate the number of topologically equivalent triangulations (definitions of what we mean for distinct coverings and distinct dynamical triangulations of a given manifold M are given in Section 2.1).

To be more precise on the meaning of topological equivalence adopted here, it must be stressed that we are actually counting equivalent triangulations having a common simple homotopy type. This latter remark may need a few words of explanation.

A good counting function of utility for simplicial quantum gravity should provide the number of geodesic ball coverings in manifolds which are piecewise-linearly (PL) equivalent. But according to the finiteness theorems recalled above, asking for such a counting function is too much. In dimension three we have not yet control on the enumeration of the homeomorphism types while in dimension four no elementary enumeration is affordable for the PL types (by Cerf's theorem we know that every PL 4-manifold carries a unique differentiable structure; there can be only countably many differentiable structures on a compact topological 4-manifolds, while there are uncountably many diffeomorphism classes of 4-manifolds homeomorphic to $\mathbb{R}^4$; in this sense counting PL structures is directly connected with the enumeration of differential structures). Thus in the physically significant dimensions there is no obvious enumerative criterion for PL structures.

The necessary compromise between what can be counted and what is of utility for quantum gravity brings into evidence a particular equivalence relation in homotopy known as *simple homotopy equivalence*. Two polyhedra are simple-homotopy equivalent if they have PL homeomorphic closed regular neighborhoods in some $\mathbb{R}^n$. This notion of topological equivalence associated with simple homotopy may seem too weak for our enumerative purposes, but as we shall see it is sufficient for providing a detailed exponential bound to the enumeration of dynamical triangulations.

It is also important to stress that even if the balls are topologically trivial (i.e., if they are contractible) the labeling associated with the use of the twisted cohomology $H^*_{\mathfrak{g}}$ is non-trivial. In such a case, $H^*_{\mathfrak{g}}$ reduces to the assignment of the flat bundle, over the corresponding ball, associated with the representation θ. If all balls are contractible, all such bundles are isomorphic, but, obviously, not canonically. Thus, $H^*_{\mathfrak{g}}$ can be still used as non-trivial labels for counting purposes.

The explicit counting of the inequivalent orbits, under permutations of the balls, associated with such colorings is obtained by means of Pólya's enumeration theorem [Bo]. More precisely, Pólya's theorem is used for counting geodesic ball packings, so as to avoid the unwieldy complications arising from the intersections of the balls when they cover the manifold. The counting is then extended by a simple argument, (relying, however on a deep compactness theorem by Gromov) to the geodesic ball coverings associated with the packings.

From this enumeration we get that, *in the given representation* θ, the number, $B_{\mathrm{Cov}}(\Delta^{\mathfrak{g}}, \lambda)$, of distinct geodesic ball coverings with λ balls that can be introduced in the manifold M is bounded above, for large λ, by

$$B_{\mathrm{Cov}}(\Delta^{\mathfrak{g}}, \lambda) \leq \frac{1}{\sqrt{2\pi}\,\Delta^{\mathfrak{g}}(M)} \sqrt{\frac{n+2}{n+1}} \left[\frac{(n+2)^{n+2}}{(n+1)^{n+1}} \tilde{w} \right]^{\lambda} \lambda^{-1/2} \left(1 + O(\lambda^{-3/2})\right), \quad (3)$$

where n denotes the dimension of M, $\Delta^{\mathfrak{g}}(M)$ is the Reidemeister torsion of M in the *given representation* $\theta : \pi_1(M) \to G$, and where $\tilde{w}$ is, roughly speaking, the Reidemeister torsion of the *dominant* twisted cohomology group of the balls.

Recall that, given a manifold M and a representation of its fundamental group $\pi_1(M)$ in a flat bundle $\mathfrak{g}_\theta$, the Reidemeister torsion is a generalized volume element constructed from the twisted cohomology groups $H^i(M, \mathfrak{g}_\theta)$. In even dimension, if M is compact, orientable, and without boundary, it can be shown by Poincaré duality that $\Delta^{\mathfrak{g}}(M) = 1$. However, this latter result does not hold for the balls of the covering since they have a boundary. In such a case, the corresponding torsion $\tilde{w}$ depends non-trivially on the metric of the ball, too.

Topologically speaking, (3) is estimating the number of geodesic ball coverings on a manifold of given *simple homotopy* type (for a given $\pi_1(M)$ and a given representation θ, this simple homotopy type is characterized by the torsion). If one is interested in counting coverings (and triangulations) *just* on a manifold of given fundamental group, then (3) reduces to

$$\frac{1}{\sqrt{2\pi}} \sqrt{\frac{n+2}{n+1}} \left[\frac{(n+2)^{n+2}}{(n+1)^{n+1}} \right]^{\lambda} \lambda^{-1/2} \left(1 + \mathrm{O}(\lambda^{-3/2})\right), \quad (4)$$

which does not depend any longer on the representation $\theta : \pi_1(M) \to G$, and provides a significant exponential bound to the number of distinct coverings that one can introduce on M.

In particular, the number of distinct geodesic ball coverings, with λ balls, that can be introduced on a surface Σ of given topology turns out to be asymptotically bounded by

$$\frac{2}{\sqrt{6\pi}} \left[\frac{4^4}{3^3} \right]^{\lambda} \lambda^{-1/2}. \quad (5)$$

This bound is perfectly consistent with the classical result provided by W. Tutte [BIZ] according to which the number of distinct triangulations, with λ vertices, of a surface (with the topology of the sphere) is asymptotically

$$\frac{1}{64\sqrt{6\pi}} \left[\frac{4^4}{3^3} \right]^{\lambda} \lambda^{-7/2}. \quad (6)$$

The finer entropy estimates (3) do depend on the particular representation θ, thus a more interesting object to discuss is their average over all possible inequivalent representations in the given group G obtained by integrating (3) over the representation variety $\mathrm{Hom}(\pi_1(M), G)/G$.

Entropy estimates at fixed λ, and $n = 2$. Denoting by θ the dominant representations (in a formal saddle point evaluation of the integral over inequivalent representations) we get for the entropy estimate, up to some inessential constants

$$\int\limits_{\mathrm{Hom}(\pi_1(M),G)/G} B_{\mathrm{Cov}}(\Delta^{\mathfrak{g}}, \lambda)$$
$$\leq \sum_{\theta\in\mathrm{Hom}_0} \frac{2}{\sqrt{6\pi}\,\Delta^{\mathfrak{g}}_{\theta}(M)} \left[\frac{4^4}{3^3}\tilde{w}_{\theta}\right]^{\lambda} \lambda^{[-((h-1)/2)\dim(G)-(\dim(z(\theta))/2-1/2]}(1+\cdots), \tag{7}$$

where Hom_0 denotes the (finite) set of representations contributing to the saddle point evaluation, h denotes the genus of the surface M, and $z(\theta)$ denotes the centralizer of $\theta(\pi_1(M))$ in the Lie group G.

We define the critical exponent $\eta(G)$ associated with the entropy function $B_{\mathrm{Cov}}(\Delta^{\mathfrak{g}}, \lambda)$ by means of the relation

$$\int\limits_{\mathrm{Hom}(\pi_1(M),G)/G} B_{\mathrm{Cov}}(\Delta^{\mathfrak{g}}, \lambda) \equiv \mathrm{Meas}\left(\frac{\mathrm{Hom}(\pi_1(M),G)}{G}\right)\exp[c\lambda]\lambda^{\eta_{\mathrm{sup}}-3}, \tag{8}$$

where c is a suitable constant (depending on G). Then (7) provides also a bound for $\eta(G)$ given by (for a given $\theta \in \mathrm{Hom}_0$)

$$\eta(G) \leq 2 + (1-h)\tfrac{1}{2}(\dim(G)) + \tfrac{1}{2}(1-\dim(z(\theta))). \tag{9}$$

For instance, for $G = U(1)$, we get

$$\eta(G) \leq 2 + \tfrac{1}{2}(1-h), \tag{10}$$

which is consistent with KPZ scaling. This bound is an equality in the obvious case $h = 1$, while it is sharp in the remaining cases. It is likely that (9) holds also in the case where there is a strong coupling of 2D-gravity with matter, namely in the regime where KPZ scaling breaks down.

Entropy estimates at fixed λ, and $n = 4$. In the four-dimensional case we obtain, again through a formal saddle point evaluation, and up to some inessential factors

$$\sum_{\theta\in\mathrm{Hom}_0} \frac{\sqrt{6}}{\sqrt{10\pi}\,\Delta^{\mathfrak{g}}_{\theta}(M)} \left[\frac{6^6}{5^5}\tilde{w}_{\theta}\right]^{\lambda} \lambda^{[\dim(G)\chi(M)/8-b(2)/8-1/2]}(1+\cdots), \tag{11}$$

where $\chi(M)$ is the Euler–Poincaré characteristic of M and $b(2)$ is the second Betti number associated with $H^*_{\mathfrak{g}}(M)$.

Notice that in the above expressions we can set $\Delta^{\mathfrak{g}}_{\theta}(\mathcal{O}_h) = 1$ (the torsion being trivial in even dimensions for a closed, orientable manifold). The bound on the critical exponent corresponding to this entropy estimate is (for a given $\theta \in \mathrm{Hom}_0$)

$$\eta(G) \leq \tfrac{5}{2} + \tfrac{1}{8}\dim(G)\chi(M) - \tfrac{1}{8}b(2). \tag{12}$$

This exponent, evaluated for the four-sphere, takes on the value $\frac{11}{4}$ which is larger than the corresponding exponent obtained through numerical simulations (see e.g., [Va]). In this latter case, the available values of this exponent are typically affected by a large uncertainty.

Nonetheless, numerical evidence seems to indicate a rough value around the figures 0.40, 0.57, thus our bound is strict and likely not optimal.

We are perfectly aware that this work is incomplete in many respects. In particular, it is annoying that one does not get an entropy estimate directly for triangulated four-manifolds but rather for geodesic ball covered manifolds. However, this estimate is sufficient for controlling the number of topologically (in the simple-homotopical sense) equivalent dynamical triangulations on four-manifolds of bounded geometry, and it is, we believe, a good starting point for a further understanding of discrete models of four-dimensional quantum gravity. (See also note added in proof at the end.)

We now turn to a more extensive discussion of our subject.

2. Metric ball coverings and triangulated manifolds

As recalled in the introductory remarks, in order to regain a smooth geometric perspective when dealing with a dynamically triangulated manifold $\mathcal{T}$, we have to move our observation point far away from $\mathcal{T}$ (for rather different reasons this same point of view, which is the essence of a scaling limit, is advocated in geometric group theory [Gr2]). In this way, and under suitable rescaling for the coupling constants of the theory, the details of the triangulation $\mathcal{T}$ may fade away at criticality, and the simplices of $\mathcal{T}$ coalesce into extended objects, generalized metric manifolds representing the *space–time* manifolds (or more correctly, an Euclidean version of them) dominating the statistical sum of the model considered.

Technically speaking, this limiting procedure appeals here to a topology in the set of metric spaces coming along with a Hausdorff-type metric. This was rather explicitly suggested in 1981 by Fröhlich [Fro] in his unpublished notes on Regge's model. For completely different reasons, and more or less in the same period, this notion of topology was made precise by Gromov [Gr1], and used by him very effectively to discuss the compactness properties of the space of riemannian structures. A detailed analysis is presented in [Gr1,CM2], and instead of repeating it here we give the intuition and a few basic definitions. The rough idea is that given a length cut-off ϵ, two riemannian manifolds are to be considered near in this topology (one is the ϵ-Gromov–Hausdorff approximation of the other) if their metric properties are similar at length scales $L \geq \epsilon$. This intuition can be made more precise as follows.

Consider two riemannian manifolds M_1 and M_2 (or more in general any two compact metric spaces) let $d_{M_1}(\cdot,\cdot)$ and $d_{M_2}(\cdot,\cdot)$ respectively denote the corresponding distance functions, and let $\phi : M_1 \to M_2$ be a map between M_1 and M_2 (this map is not required to be continuous). If ϕ is such that: (i) the ϵ-neighborhood of $\phi(M_1)$ in M_2; is equal to M_2; and (ii) for each x, y in M_1 we have

$$|d_{M_1}(x,y) - d_{M_2}(\phi(x),\phi(y))| < \epsilon, \tag{13}$$

then ϕ is said to be an *ϵ-Hausdorff approximation*. The *Gromov–Hausdorff distance* between the two riemannian manifolds M_1 and M_2, $d_G(M_1, M_2)$, is then defined according to [Gr1].

Definition 1. $d_G(M_1, M_2)$ is the lower bound of the positive numbers ϵ such that there exist ϵ-Hausdorff approximations from M_1 to M_2 and from M_2 to M_1.

The notion of ϵ-Gromov–Hausdorff approximation is the weakest large-scale equivalence relation between metric spaces of use in geometry, and is manifestly adapted to the needs of simplicial quantum gravity (think of a manifold and of a simplicial approximation to it).

Notice that d_G is not, properly speaking, a distance since it does not satisfy the triangle inequality, but it rather gives rise to a metrizable uniform structure in which the set of isometry classes of all compact metric spaces (not just riemannian structures) is Hausdorff and complete. This enlarged space does naturally contain topological (metric) manifolds and curved polyhedra. As stressed in [Pt], the importance of this notion of distance lies not so much in the fact that we have a distance function, but in that we have a way of measuring when riemannian manifolds (or more general metric spaces) look alike.

In order to provide the entropy of four-dimensional triangulated manifolds, we need to use Gromov–Hausdorff topology quite superficially. Explicitly, it only appears in the ensemble of manifolds for which we characterize the entropy function.

Definition 2. For r a real number, D and V positive real numbers, and n a natural number, let $\mathcal{R}(n, r, D, V)$ denote the Gromov–Hausdorff closure of the space of isometry classes of closed connected n-dimensional riemannian manifolds (M, g) with sectional curvature bounded below by r, viz.,

$$\inf_{x \in M} \{\inf\{g_x(\mathrm{Riem}_x(u, v)u;\, v)\colon u, v \in T_x M, \text{orthonormal}\}\} \geq r$$

and diameter bounded above by D,

$$\mathrm{diam}(M) \equiv \sup_{(p,q) \in M \times M} d_M(p, q) \leq D$$

and volume bounded below by V.

The point in the introduction of $\mathcal{R}(n, r, D, V)$ or of more general classes of metric spaces with a lower bound on a suitably defined notion of curvature, is that for any manifold (or metric space) M in such a class one gets a packing information which is most helpful in controlling the topology in terms of the metric geometry. In the case of $\mathcal{R}(n, r, D, V)$ this packing information is provided by suitable coverings with geodesic (metric) balls yielding a coarse classification of the riemannian structures occurring in $\mathcal{R}(n, r, D, V)$ (notice that these coverings can be introduced under considerably less restrictive conditions, in particular it is sufficient to have a lower bound on the Ricci tensor, and an upper bound on the diameter, [GP]).

In order to define such coverings [GP], let us parametrize geodesics on $M \in \mathcal{R}(n, r, D, V)$ by arc length, and for any $p \in M$ let us denote by $\sigma_p(x) \equiv d_M(x, p)$ the distance function of the generic point x from the chosen point p. Recall that $\sigma_p(x)$ is a smooth function away from $\{p \cup C_p\}$, where C_p, a closed nowhere dense set of measure zero, is the cut locus of p. Recall also that a point $y \neq p$ is a critical point of $\sigma_p(x)$ if for all vectors $v \in TM_y$, there is a minimal geodesic γ from y to p such that the angle between v and $\dot{\gamma}(0)$ is not greater than $\frac{1}{2}\pi$.

Definition 3. For any manifold $M \in \mathcal{R}(n, r, D, V)$ and for any given $\epsilon > 0$, it is always possible to find an ordered set of points $\{p_1, \ldots, p_N\}$ in M, so that, [GP]

(i) the open metric balls (the *geodesic balls*) $B_M(p_i, \epsilon) = \{x \in M \mid d(x, p_i) < \epsilon\}$, $i = 1, \ldots, N$, cover M; in other words the collection

$$\{p_1, \ldots, p_N\} \tag{14}$$

is an ϵ-net in M.

(ii) the open balls $B_M(p_i, \frac{1}{2}\epsilon)$, $i = 1, \ldots, N$, are disjoint, i.e. $\{p_1, \ldots, p_N\}$ is a *minimal* ϵ-net in M.

Similarly, upon considering the higher-order intersection patterns of the set of balls $\{B_M(p_i, \epsilon)\}$, we can define the two-skeleton $\Gamma^{(2)}(M)$, and eventually the nerve $\mathcal{N}\{B_i\}$ of the geodesic balls covering of the manifold M.

Definition 4. Let $\{B_i(\epsilon)\}$ denote a minimal ϵ-net in M. The geodesic ball nerve $\mathcal{N}\{B_i\}$ associated with $\{B_i(\epsilon)\}$ is the polytope whose k-simplices $p^{(k)}_{i_1 i_2 \cdots i_{k+1}}$, $k = 0, 1, \ldots,$ are defined by the collections of $k + 1$ geodesic balls such that $B_1 \cap B_2 \cap \cdots \cap B_{k+1} \neq \emptyset$.

Thus, for instance, the vertices $p^{(0)}_i$ of $\mathcal{N}\{B_i\}$ correspond to the balls $B_i(\epsilon)$; the edges $p^{(1)}_{ij}$ correspond to pairs of geodesic balls $\{B_i(\epsilon), B_j(\epsilon)\}$ having a non-empty intersection $B_i(\epsilon) \cap B_j(\epsilon) \neq \emptyset$; and the faces $p^{(2)}_{ijk}$ correspond to triples of geodesic balls with not-empty intersection $B_i(\epsilon) \cap B_j(\epsilon) \cap B_k(\epsilon) \neq \emptyset$.

Remark 1. Notice that, in general, this polytope has a dimension which is greater than the dimension n of the underlying manifold. However, as $\epsilon \to 0$, such dimension cannot grow arbitrarily large being bounded above by a constant depending only on r, n and D (see below).

Minimal geodesic ball coverings provide a means for introducing a short-distance cut-off as for a dynamical triangulation, while hopefully mantaining a more direct connection with the geometry and in particular with the topology of the underlying manifold. The basic observation here is that such coverings are naturally labeled (or colored) by the fundamental groups of the balls. Indeed, according to the properties of the distance function (see for instance [Ch]) given $\epsilon_1 < \epsilon_2 \leq \infty$, if in $\bar{B}_i(\epsilon_2) \backslash B_i(\epsilon_1)$ there are no critical points of the distance function σ_i, then this region is homeomorphic to $\partial B_i(\epsilon_1) \times [\epsilon_1, \epsilon_2]$, and $\partial B_i(\epsilon_1)$ is a topological submanifold without boundary. One defines a *criticality radius* ϵ_i for each ball $B_i(\epsilon)$, as the largest ϵ such that $B_i(\epsilon)$ is free of critical points. Corresponding to such value of the radius ϵ, the ball $B_i(\epsilon)$ is homeomorphic to an arbitrarily small open ball with center p_i, and thus it is homeomorphic to a standard open ball. It can be easily checked, through direct examples, that the criticality radius of geodesic balls of manifolds in $\mathcal{R}(n, r, D, V)$

can be arbitrarily small (think of the geodesic balls drawn near the rounded off tip of a cone) thus arbitrarily small metric balls in manifolds of bounded geometry are not necessarily contractible, and therefore, in general, the $B_i(\epsilon)$ are not homeomorphic to a standard open ball.

2.1. Connections with dynamical triangulations

Since the geodesic ball coverings are to play an important role in our development, a few remarks about the connection between such coverings and dynamical triangulations are in order.

As recalled in the introductory remarks, a dynamical triangulation of a (pseudo)-manifold can be used to produce a metric on that manifold, by declaring all the simplices in the triangulation isometric to the standard simplex of the appropriate dimension, and by assuming that the edge lengths are all equal to some fundamental length. An n-dimensional dynamical triangulation is actually constructed by successively gluing pairs of such flat n-simplices along some of their $(n-1)$-faces, until one gets a complex without boundary. This gives a collection of compatible metrics, on pieces of the resulting pseudo-manifold, which can be extended to a genuine metric, since between any two points there is a path minimizing the distance (one speaks of a pseudo-manifold since, for $n > 2$, the complex constructed by this gluing procedure may have some vertices whose neighborhood is not homeomorphic to the standard Euclidean ball).

We identify two dynamical triangulations of the same underlying manifold M if there is a one-to-one mapping of vertices, edges, faces, and higher-dimensional simplices of one onto vertices, edges, faces, and higher-dimensional simplices of the other which preserves incidence relations. If no such mapping exists the the two dynamical triangulations are said to be *distinct*. Notice that sometimes one says that such dynamical triangulations are combinatorially distinct. Since this may be source of confusion (in dynamical triangulation theory the notion of combinatorial equivalence is synonimus of PL-equivalence, see below) we carefully avoid the use of the qualifier "combinatorial" in this context.

On a dynamical triangulation so constructed, one can define metric balls, and consider minimal geodesic ball coverings. Actually, it is clear that in a generic metric space there are many distinct ways of introducing minimal geodesic ball coverings with a given radius of the balls. As a simple example, consider a portion of Euclidean three-space (one may wish to identify boundaries so to obtain a flat three-torus). It is well known that a portion of Euclidean three-space can be packed and covered, with small spheres of a given radius, in many inequivalent ways, to the effect that in the limit, for $\mathbb{R}^3$, there are uncountably many such coverings.

As for dynamical triangulations, we identify two geodesic ball coverings, $\{B_i\}_1$ and $\{B_k\}_2$, of the same underlying manifold M if there is a one-to-one mapping of vertices, edges, faces, and higher-dimensional simplices of the nerve of $\{B_i\}_1$ onto vertices, edges, faces, and higher-dimensional simplices of the nerve of $\{B_k\}_2$, which preserves incidence relations. If no such mapping exists the two geodesic ball coverings are said to be *distinct*.

Generally, given a manifold triangulated with n-dimensional simplices with a given edge length, we can always introduce a minimal geodesic ball covering whose properties are closely connected with the properties of the underlying triangulation. This can be done according to the following definition.

Definition 5. Let (M, T) denote a manifold (a compact polyhedron) triangulated with fixed edge length equilateral simplices, and let ϵ denote the length of the edges. With each vertex p_i belonging to the triangulation, we associate the largest open metric ball contained in the open star of p_i. Then the metric ball covering of (M, T) generated by such balls $\{B_i\}$ is a minimal geodesic ball covering. It defines the geodesic ball covering associated with the dynamically triangulated manifold (M, T).

It is immediate to see that the set of balls considered defines indeed a minimal geodesic ball covering. The open balls obtained from $\{B_i\}$ by halfing their radius are disjoint being contained in the open stars of $\{p_i\}$ in the baricentric subdivision of the triangulation. The balls with doubled radius cover (M, T), since they are the largest open balls contained in the stars of the vertices $\{p_i\}$ of T.

In order to connect the enumeration of *distinct* geodesic ball coverings with the enumeration of *distinct* triangulations, we recall that any two dynamical triangulations are said to be *Combinatorially Equivalent* if the two triangulations can be subdivided into the same finer triangulation. In other words, if they correspond to triangulations T_1 and T_2 of the same abstract compact polyhedron P. This last remark follows since any two triangulations of a compact polyhedron have a common subdivision. Notice that quite often, when considering a particular triangulation (M, T) is standard usage to identify the abstract polyhedron M with $|T|$, the union of the cells of T (the underlying polyhedron associated with T). The more so when dealing with dynamical triangulations, where the emphasis is on the actual construction of T. This identification is a source of confusion in enumerative problems and we shall keep distinct the abstract polyhedron M from $|T|$.

The relation between a dynamically triangulated manifold and the associated geodesic ball covering implies the following lemma.

Lemma 1. *If (M, T_1) and (M, T_2) are any two distinct combinatorially equivalent fixed edge-length triangulations, then the corresponding geodesic ball coverings $\{B_i\}_1$ and $\{B_i\}_2$ are distinct.*

Proof. This amounts to prove that the nerve associated with geodesic ball covering corresponding to a fixed edge-length triangulation is isomorphic, as a simplicial complex, to the given triangulation. If this were not the case, then, there should be at least one k-simplex in the nerve, $p^k_{i_1\cdots i_{k+1}}$, associated with the mutual intersections of $k+1$ balls, for some $k > 1$, such that the vertices of such k-simplex correspond to vertices of the triangulation not connected by links. But then the corresponding balls B_i cannot mutually intersect, since they are contained in the disjoint open stars of the respective vertices. Thus, there cannot be such a simplex $p^k_{i_1\cdots i_{k+1}}$ to begin with. □

In general, by choosing a different prescription for geodesic ball coverings associated with fixed edge-length triangulations (e.g., by choosing differently the centers of the balls B_i) we get a nerve which is not necessarily isomorphic to the dynamical triangulation itself. And, as already stressed, the dimension of the nerve is, in general, larger than the dimension of the underlying manifold, and even if we restrict our attention, say, to the four-skeleton, we get a complex which is not the triangulation of a four-manifold.

If we combine this remark with Lemma 1 then we get the following proposition.

Proposition 1. *For a given minimal short-distance cut-off* ϵ *the number of distinct geodesic ball coverings is not smaller than the number of corresponding dynamical triangulations.*

Incidentally, by means of the above construction of a geodesic ball covering associated with a given fixed edge-length triangulation, we can also explain, in terms of dynamical triangulations, the origin of the possible non-trivial topology of the balls.

Recall that in an n-dimensional simplicial manifold each vertex has a sufficiently small neighborhood which is homeomorphic to the standard n-dimensional Euclidean ball. And, in such a case, the above minimal geodesic ball covering is necessarily generated by contractible balls. Thus, non-contractible balls are present if we allow for dynamical triangulations associated with simplicial pseudo-manifolds. And this is the typical case, at least in dimension $n > 2$, since pseudo-manifolds are the natural outcome of the process of gluing n-simplices along their $(n-1)$-faces.

2.2. Homotopy and geodesic ball coverings

The above remarks suggest that one should be careful in understanding in what sense, for ϵ sufficiently small, the geodesic ball nerve gives rise to a polytope whose topology approximates the topology of the manifold $M \in \mathcal{R}(n, r, D, V)$. This is a natural consequence of the fact that the criticality radius for the geodesic balls is not bounded below. In full generality, the geodesic ball nerve controls only the homotopy type of the manifold [GPW]. This follows by noticing that the inclusion of sufficiently small geodesic balls into suitably larger balls is homotopically trivial, and the geodesic ball nerve is thus a polytope which is *homotopically dominating* the underlying manifold, viz., there exist maps $f : M \to \mathcal{N}(B_i)$, and $g : \mathcal{N}(B_i) \to M$, with $g \cdot f$ homotopic to the identity mapping in M.

It may appear rather surprising, but this homotopical control is more than sufficient for yielding the entropic estimates we are looking for.

On the geometrical side, there are a wealth of good properties of geodesic ball coverings which make them particularly appealing for applications in simplicial quantum gravity. As a good start, we can notice that the equivalence relation defined by manifolds with (combinatorially) isomorphic geodesic ball one-skeletons partitions $\mathcal{R}(n, r, D, V)$ into disjoint equivalence classes whose finite number can be estimated in terms of the parameters n, k, D. Each equivalence class of manifolds is characterized by the abstract (unlabeled) graph $\Gamma_{(\epsilon)}$ defined by the 1-skeleton of the $L(\epsilon)$-covering. The order of any such graph (i.e., the number of vertices) defines the *filling function* $N^{(0)}_{(\epsilon)}$, while the structure of the edge set of $\Gamma_{(\epsilon)}$ defines the (first-order) intersection pattern $I_{(\epsilon)}(M)$ of $(M, \{B_i(\epsilon)\})$.

It is important to remark that on $\mathcal{R}(n, r, D, V)$ neither the filling function nor the intersection pattern can be arbitrary. The filling function is always bounded above for each given ϵ, and the best filling, with geodesic balls of radius ϵ, of a riemannian manifold of diameter $\mathrm{diam}(M)$, and Ricci curvature $\mathrm{Ric}(M) \geq (n-1)H$, is controlled by the corresponding filling of the geodesic ball of radius $\mathrm{diam}(M)$ on the space form of constant curvature given by H, the bound being of the form $N^{(0)}_{\epsilon} \leq N(n, H(\mathrm{diam}(M))^2, (\mathrm{diam}(M))/\epsilon)$ [Gr1].

The multiplicity of the first intersection pattern is similarly controlled through the geometry of the manifold to the effect that the average degree $d(\Gamma)$ of the graph $\Gamma_{(\epsilon)}$ (i.e., the average number of edges incident on a vertex of the graph) is bounded above by a constant as the radius of the balls defining the covering tend to zero (i.e., as $\epsilon \to 0$). Such constant is independent from ϵ and can be estimated [Gp] in terms of the parameters n, and $H(\mathrm{diam}(M))^2$ (it is this boundedness of the order of the geodesic ball coverings that allows for the control of the dimension of the geodesic ball nerve).

As expected, the filling function can be also related to the volume $v = \mathrm{Vol}(M)$ of the underlying manifold M. This follows by noticing that [Zh] for any manifold $M \in \mathcal{R}(n, r, D, V)$ there exist constants C_1 and C_2, depending only on n, r, D, V, such that, for any $p \in M$, we have

$$C_1\epsilon^n \leq \mathrm{Vol}(B_\epsilon(p)) \leq C_2\epsilon^n, \tag{15}$$

with $0 \leq \epsilon \leq D$ (actually, here and in the previous statements a lower bound on the Ricci curvature suffices). Explicitly, the constants C_1 and C_2 are provided by

$$C_1 \equiv \frac{V}{\mathrm{Vol}^r(B(D))} \inf_{0 \geq \epsilon \geq D} \frac{1}{\epsilon^n} \int_0^\epsilon \left(\frac{\sinh\sqrt{-r}t}{\sqrt{-r}}\right)^{n-1} \mathrm{d}t \tag{16}$$

and

$$C_2 \equiv \sup_{0 \geq \epsilon \geq D} \frac{1}{\epsilon^n} \int_0^\epsilon \left(\frac{\sinh\sqrt{-r}t}{\sqrt{-r}}\right)^{n-1} \mathrm{d}t, \tag{17}$$

where $\mathrm{Vol}^r(B(D))$ denotes the volume of the geodesic ball of radius D in the (simply connected) space of constant curvature $-r$, and D, r, V, n are the parameters characterizing the space of bounded geometries $\mathcal{R}(n, r, D, V)$ under consideration.

Thus, if v is the given volume of the underlying manifold M, by the Bishop–Gromov relative comparison volume theorem we obtain that there exists a function $\rho_1(M)$, depending on n, r, D, V, and on the actual geometry of the manifold M, with $C_1 \leq (\rho_1(M))^{-1} \leq C_2$, and such that, for any $m \geq m_0$, we can write

$$N^{(0)}_\epsilon(M) = v\rho_1(M)\epsilon^{-n}. \tag{18}$$

We conclude this section by recalling the following basic finiteness results. They provide the topological rationale underlying the use of spaces of bounded geometries in simplicial quantum gravity. We start with a result expressing finiteness of homotopy types of manifolds of bounded geometry [GPW].

Theorem 1. *For any dimension $n \geq 2$, and for ϵ sufficiently small, manifolds in $\mathcal{R}(n, r, D, V)$ with the same geodesic ball 1-skeleton $\Gamma_{(\epsilon)}$ are homotopically equivalent, and the number of different homotopy-types of manifolds realized in $\mathcal{R}(n, r, D, V)$ is finite and is a function of n, $V^{-1}D^n$ and rD^2.*

(Two manifolds M_1 and M_2 are said to have the same homotopy type if there exists a continuous map ϕ of M_1 into M_2 and f of M_2 into M_1, such that both $f \cdot \phi$ and $\phi \cdot f$ are homotopic to the respective identity mappings, I_{M_1} and I_{M_2}. Obviously, two homeomorphic manifolds are of the same homotopy type, but the converse is not true.)

Notice that in dimension three one can replace the lower bound of the sectional curvatures with a lower bound on the Ricci curvature [Zh]. Actually, a more general topological finiteness theorem can be stated under a rather weak condition of local geometric contractibility. Recall that a continuous function $\psi : [0, \alpha) \to \mathbb{R}^+$, $\alpha > 0$, with $\psi(0) = 0$, and $\psi(\epsilon) \geq \epsilon$, for all $\epsilon \in [0, \alpha)$, is a local geometric contractibility function for a riemannian manifold M if, for each $x \in M$ and $\epsilon \in (0, \alpha)$, the open ball $B(x, \epsilon)$ is contractible in $B(x, \psi(\epsilon))$ [GrP] (which says that a small ball is contractible relative to a bigger ball). Given a local geometric contractibility function one obtains the following [GrP].

Theorem 2. *Let $\psi : [0, \alpha) \to \mathbb{R}^+$, $\alpha > 0$, be a continuous function with $\psi(\epsilon) \geq \epsilon$ for all $\epsilon \in [0, \alpha)$ and such that, for some constants C and $k \in (0, 1]$, we have the growth condition $\psi(\epsilon) \leq C\epsilon^k$, for all $\epsilon \in [0, \alpha)$. Then for each $V_0 > 0$ and $n \in \mathbb{R}^+$ the class $\mathcal{C}(\psi, V_0, n)$ of all compact n-dimensional Riemannian manifolds with volume $\leq V_0$ and with ψ as a local geometric contractibility function contains:*

(i) *finitely many simple homotopy types (all n),*
(ii) *finitely many homeomorphism types if $n = 4$,*
(iii) *finitely many diffeomorphism types if $n = 2$ or $n \geq 5$.*

Actually the growth condition on ψ is necessary in order to control the dimension of the limit spaces resulting from Gromov–Hausdorff convergence of a sequence of manifolds in $\mathcal{C}(\psi, V_0, n)$. As far as homeomorphism types are concerned, this condition can be removed [Fe]. Note moreover that infinite-dimensional limit spaces cannot occur in the presence of a lower bound on sectional curvature as for manifolds in $\mathcal{R}(n, r, D, V)$. Finiteness of the homeomorphism types cannot be proved in dimension $n = 3$ as long as the Poincaré conjecture is not proved. If there were a fake three-sphere then one could prove [Fe] that a statement such as (ii) above is false for $n = 3$. Finally, the statement on finiteness of *simple homotopy* types, in any dimension, is particularly important for the applications in quantum gravity we discuss in the sequel. Roughly speaking, the notion of simple homotopy is a refinement of the notion of homotopy equivalence, and it may be thought of as an intermediate step between homotopy equivalence and homeomorphism.

The machinery needed to characterize the entropy function for geodesic ball coverings in four-dimensional manifolds of bounded geometry is now at hand.

3. Topology and entropy of metric ball coverings

The combinatorial structure associated with minimal geodesic ball coverings appears more complex than the combinatorial structure of dynamical triangulations. However, the counting of all possible distinct minimal geodesic ball coverings of given radius, on a manifold of bounded geometry, is more accessible than the counting of distinct dynamical triangulations.

This fortunate situation arises because we can label the balls $B_\epsilon(p_i)$ with their non-trivial fundamental group $\pi_1(B_\epsilon(p_i); p_i)$ (obviously, since we are interested to the distinct classes of covering, we need to factor out the trivial labeling associated with the centers p_i of the balls). Thus the counting problem we face is basically the enumeration of such inequivalent topological labelings of the balls of the covering. Such an enumeration is not yet very accessible. As it stands, there are constraints on the fundamental groups $\pi_1(B_\epsilon(p_i); p_i)$, expressed by Seifert–VanKampen's theorem, which express the match between the intersection pattern of the balls and the homomorphisms $\pi_1(\cap_i B_\epsilon(p_i); x_0) \rightarrow \pi_1(M; x_0)$ associated with the injection of clusters of mutually intersecting balls into M (x_0 being a base point in $M \cap_i B_\epsilon(p_i)$). Such difficulties can be circumvented by using as labels, rather than the fundamental groups themselves, a cohomology with local coefficients in representations of $\pi_1(B_\epsilon(p_i); p_i)$ into a Lie group G. Roughly speaking, this means that we are using *flat bundles corresponding to the representation* θ as non-trivial labels for the balls.

This construction gives to the counting problem of inequivalent geodesic ball coverings an unexpected interdisciplinary flavor which blends in a nice way riemannian geometry (*the metric properties of the balls*), topology (*the action of the fundamental group on homology*), and representation theory (*the structure of the space of inequivalent representation of the fundamental group of an n-dimensional manifold,* $n \geq 3$), into a Lie group G.

We wish to stress that a similar approach may be suitable also for a direct enumeration of dynamical triangulations since flat bundles on the simplexes (again associated with representations of the fundamental group of the underlying PL-manifold) do provide a natural topological labeling of the simplexes. It is true that such simplexes have no non-trivial topology (they are contractible) and that a flat bundle (associated with the given representation θ) over one simplex is isomorphic to the flat bundle over any other simplex. However, such isomorphism is not canonical, as is obviously shown by the fact that one may get a non-trivial flat bundle by gluing such local bundles if the underlying manifold has a non-trivial fundamental group (we wish to thank Ambjørn and Durhuus for discussions that draw our attention to this further possibility).

3.1. Cohomology with local coefficients and representation spaces

In order to describe either the topological aspects or the basic properties of the representation spaces mentioned above and which play a prominent role into our entropy estimate, it will be convenient to recall some basic facts about cohomology with local coefficients. Details of the theory summarized here can be found in [DNF,RS,JW].

Let $(M, \{B_\epsilon(p_i)\}) \in \mathcal{R}(n, r, D, V)$ be a manifold of bounded geometry endowed with a minimal geodesic ball covering, and thought of as a cellular or simplicial complex (for instance by associating with $(M, \{B_\epsilon(p_i)\})$ the corresponding nerve $\mathcal{N}$; in what follows we tacitly exploit the fact that a sufficiently fine minimal geodesic ball covering has the same homotopy type of the underlying manifold). We let $\pi_1(M)$ denote the fundamental group of $(M, \{B_\epsilon(p_i)\})$. Such $\pi_1(M)$ is finitely generated, and can be assumed to be finitely presented.

Let $\hat{M} \to M$ denote the universal covering of M, on which $\pi_1(M)$ acts by deck transformations. Let us introduce the homology complex $C_*(\hat{M}) = \bigoplus_{i \in \mathbb{N}} C_i(\hat{M})$ where the chains in $C_i(\hat{M})$ are of the form $\sum_{j,\gamma} \lambda_{j\gamma} a_j(\hat{\sigma}^i_\gamma)$ where $\lambda_{j\gamma}$ are integers, $a \in \pi_1(M)$, and $\hat{\sigma}^i_\gamma$ are a set of chosen i-cells in $\hat{M}$. This is tantamount to say that the chains $C_i(\hat{M})$ have coefficient in the group ring $\mathbb{Z}\pi_1(M)$, i.e., in the set of all finite formal sums $\sum n_i a_i$, $n_i \in \mathbb{Z}$, $a_i \in \pi_1(M)$, with the natural definition of addition and multiplication.

Let $\theta : \pi_1(M) \to G$, be a representation of $\pi_1(M)$ in a Lie group G whose Lie algebra $\mathfrak{g}$ carries an Ad-invariant, symmetric, non-degenerate bilinear form (i.e., a metric). The representation θ defines a flat bundle, that we denote by $\mathfrak{g}_\theta$. This bundle is costructed by exploiting the adjoint representation of G on its Lie algebra $\mathfrak{g}$, i.e., $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$, and by considering the action of $\pi_1(M)$ on $\mathfrak{g}$ generated by composing the adjoint action and the representation θ:

$$\mathfrak{g}_\theta = \hat{M} \times \mathfrak{g}/\pi_1 \otimes [\mathrm{Ad}(\theta(\cdot))]^{-1}, \tag{19}$$

where $\pi_1 \otimes [\mathrm{Ad}(\theta(\cdot))]^{-1}$ acts, through $\pi_1(M)$, by deck transformations on $\hat{M}$ and by $[\mathrm{Ad}(\theta(\cdot))]^{-1}$ on the Lie algebra $\mathfrak{g}$. More explicitly, if $\hat{\sigma}_1, \hat{\sigma}_2$ are cells in $\hat{M}$, and g_1, g_2 are elements of $\mathfrak{g}$, then

$$(\hat{\sigma}_1, g_1) \sim (\hat{\sigma}_2, g_2) \tag{20}$$

if and only if

$$\hat{\sigma}_2 = \hat{\sigma}_1 a \tag{21}$$

and

$$g_2 = [\mathrm{Ad}(\theta(a))]^{-1} g_1 \tag{22}$$

for some $a \in \pi_1(M)$.

In this way we can define a cellular chain complex $C_*(M, \mathfrak{g}_\theta)$ with coefficients in the flat bundle $\mathfrak{g}_\theta$. First we consider chains with coefficients in the Lie algebra $\mathfrak{g}$, viz.,

$$\sum_j g_j \hat{\sigma}_j^i \tag{23}$$

with $g_j \in \mathfrak{g}$, and then quotient the resulting chain complex $C(\hat{M}) \otimes \mathfrak{g}$ by the action of $\pi_1 \otimes [\mathrm{Ad}(\theta(\cdot))]^{-1}$. This yields for an action of $\pi_1(M)$ on the above chains expressed by

$$a\left(\sum_j g_j \hat{\sigma}_j^i\right) \to \sum_j ([\mathrm{Ad}(\theta(a))]^{-1} g_j) a(\hat{\sigma}_j^i) \tag{24}$$

for any $a \in \pi_1(M)$ (i.e., we are considering $\mathfrak{g}$ as a $\pi_1(M)$-module). This action commutes with the boundary operator, and as a consequence of the definition of the flat bundle $\mathfrak{g}_\theta$, the resulting homology $H_*(M, \mathfrak{g}_\theta)$ can be thought of as a homology with local coefficients in the flat bundle $\mathfrak{g}_\theta$. By dualizing one defines the cohomology $H^*(M, \mathfrak{g}_\theta)$, which enjoys the usual properties of a cohomology theory. Sometimes, for ease of notation, we shall denote $H_*(M, \mathfrak{g}_\theta)$ and $H^*(M, \mathfrak{g}_\theta)$ by $H_*^{\mathfrak{g}}(M)$ and $H_{\mathfrak{g}}^*(M)$, respectively.

Let $B_\epsilon(p_h)$ be the generic ball of the covering $(M, \{B_\epsilon(p_i)\})$. If we denote by $\phi_h : \pi_1(B_\epsilon(p_h); p_h) \to \pi_1(M; p_h)$ the homomorphism induced by the obvious inclusion map, then together with θ, we may also consider the representations

$$\theta_h : \pi_1(B_\epsilon(p_h); p_h) \to \pi_1(M; p_h) \to G \tag{25}$$

obtained by composing θ with the various homomorphisms ϕ_h associated with the balls of the covering.

Notice that since arbitrarily small metric balls in manifolds $M \in \mathcal{R}(n, r, D, V)$ can be topologically rather complicated, it cannot be excluded a priori that the image $\phi_h[\pi_1(B_\epsilon(p_h); p_h)]$ in $\pi_1(M)$ (or more generally in the fundamental group of a larger, concentric ball) has an infinite number of generators. However, as follows from a result of Zhu [Zh], in order to avoid such troubles it is sufficient to choose the radius of the balls small enough.

Theorem 3. *There are constants R_0, ϵ_0 and C depending only on n, r, D, V, such that for any manifold $M \in \mathcal{R}(n, r, D, V)$, $p \in M$, $\epsilon \geq \epsilon_0$, if $i : B_\epsilon(p) \to B_{R_0\epsilon}(p)$ is the inclusion, then any subgroup K of $\phi_i(\pi_1(B_\epsilon(p)))$ satisfies $order(K) \leq C$.*

Thus in particular, there is no element of infinite order in $\phi_i(\pi_1(B_\epsilon(p)))$ whenever $\epsilon \geq \epsilon_0$.

According to this latter result, by chosing $\epsilon \geq \epsilon_0$ and by using the representations θ_h, we may define the cohomologies $H_{\mathfrak{g}}^*(B_\epsilon(p_h))$ with local coefficients in the corresponding flat bundles $\mathfrak{g}_\theta|(B_\epsilon(p_h))$ defined over the balls $B_\epsilon(p_h)$. As labels, these cohomology groups are easier to handle than the fundamental groups $\pi_1(B_\epsilon(p_i))$. This is so because the constraints we have to implement on the intersections of the balls, relating $\{H_{\mathfrak{g}}^*(B_\epsilon(p_i))\}_i$ to $H_{\mathfrak{g}}^*(M)$, are simply obtained by iterating the cohomology long exact Mayer–Vietoris sequence obtained from the short exact sequences connecting the cochains $C_{\mathfrak{g}}^*(B_\epsilon(p_i))$, $C_{\mathfrak{g}}^*(\cup B_\epsilon(p_i))$, and $C_{\mathfrak{g}}^*(\cap B_\epsilon(p_i))$. For instance, given any two intersecting balls $B(p_i)$ and $B(p_h)$, we get

$$\begin{aligned} 0 &\to C_{\mathfrak{g}}^j(B(p_i) \cup B(p_h)) \to C_{\mathfrak{g}}^j(B(p_i)) \oplus C_{\mathfrak{g}}^j(B(p_h)) \\ &\to C_{\mathfrak{g}}^j(B(p_i) \cap B(p_h)) \to 0, \end{aligned} \tag{26}$$

whose corresponding cohomology long exact sequence reads

$$\begin{aligned}\cdots &\to H^j_{\mathfrak{g}}(B(p_i)\cup B(p_h)) \to H^j_{\mathfrak{g}}(B(p_i)) \oplus H^j_{\mathfrak{g}}(B(p_h)) \to \\ &\to H^j_{\mathfrak{g}}(B(p_i)\cap B(p_h)) \to H^{j+1}_{\mathfrak{g}}(B(p_i)\cup B(p_h)) \to \cdots\end{aligned} \tag{27}$$

Similar expressions can be worked out for any cluster $\{B_\epsilon(p_i)\}_{i=1,2,\ldots}$ of intersecting geodesic balls (see Section 3.3) and they can be put at work for our counting purposes by introducing the Reidemeister torsion, a graded version of the absolute value of the determinant of an isomorphism of vector spaces.

3.2. Torsions

Let us start by recalling that by hypothesis $\mathfrak{g}$ is endowed with an Ad-invariant, symmetric, non-degenerate bilinear form (i.e., with a metric), thus we can introduce orthonormal bases, $\{X_k\}_{k=1,\ldots,\dim(G)}$, for the Lie algebra $\mathfrak{g}$. Since the adjoint representation is an orthogonal representation of G on $\mathfrak{g}$, we can introduce a volume element on the cochain complex $C^*(M,\mathfrak{g}_\theta)$, by exploiting such orthonormal bases: by identifying $C^i(M,\mathfrak{g}_\theta)$ with a direct sum of a copy of $\mathfrak{g}$ for each i-cell $\hat{\sigma}^i_j$ in $\hat{M}$, we can take $\hat{\sigma}^i_j \otimes X_k$ as an orthonormal basis of $C^i(M,\mathfrak{g}_\theta)$ and define the space of volume forms as the determinant line $\mathrm{detline}|C^*(M,\mathfrak{g}_\theta)| \equiv \prod_i(\mathrm{detline}|C^i(M,\mathfrak{g}_\theta)|)^{(-1)^i}$, where $\mathrm{detline}|C^i(M,\mathfrak{g}_\theta)|$ denotes the line of volume elements on $C^i(M,\mathfrak{g}_\theta)$ generated by all possible choices of the orthonormal bases $\hat{\sigma}^i_j \otimes X_k$. Explicitly, if on each $C^i(M,\mathfrak{g}_\theta)$ we chose the volume forms t_i, then the corresponding volume element is obtained by setting

$$t(\mathfrak{g}_\theta) = \prod_i (t_i)^{(-1)^i} \in \mathrm{detline}|C^*(M,\mathfrak{g}_\theta)|. \tag{28}$$

Let $d_i : C^{\mathfrak{g}}_i \to C^{\mathfrak{g}}_{i+1}$ be the coboundary operator in $C^*_{\mathfrak{g}}$, and as usual let us denote by $Z^i_{\mathfrak{g}} \equiv \ker(d)$, $B^i_{\mathfrak{g}} \equiv \mathrm{Im}(d)$. From the short exact sequences

$$0 \to Z^i_{\mathfrak{g}} \to C^i_{\mathfrak{g}} \to B^{i+1}_{\mathfrak{g}} \to 0 \tag{29}$$

and

$$0 \to B^i_{\mathfrak{g}} \to Z^i_{\mathfrak{g}} \to H^i_{\mathfrak{g}} \to 0 \tag{30}$$

we respectively get that there are natural isomorphisms

$$\Lambda^{\dim(Z^i)} Z^i_{\mathfrak{g}} \otimes \Lambda^{\dim(B^{i+1})} B^{i+1}_{\mathfrak{g}} \to \Lambda^{\dim(C^i)} C^i_{\mathfrak{g}} \tag{31}$$

and

$$\Lambda^{\dim(B^i)} B^i_{\mathfrak{g}} \otimes \Lambda^{\dim(H^i)} H^i_{\mathfrak{g}} \to \Lambda^{\dim(Z^i)} Z^i_{\mathfrak{g}}, \tag{32}$$

where $\Lambda^{\dim(\cdot)}$ denotes the top-dimensional exterior power on the vector space considered (recall that if $\cdots \to V_i \to V_{i+1} \to \cdots$ is a finite exact sequence, then there is a canonical isomorphism $\otimes_{i-\text{even}} \Lambda^{\dim(V_i)} V_i = \otimes_{i-\text{odd}} \Lambda^{\dim(V_i)} V_i$). It follows that there is an isomorphism

$$\Lambda^{\dim(B^i)} B^i_{\mathfrak{g}} \otimes \Lambda^{\dim(H^i)} H^i_{\mathfrak{g}} \otimes \Lambda^{\dim(B^{i+1})} B^{i+1}_{\mathfrak{g}} \to \Lambda^{\dim(C^i)} C^i_{\mathfrak{g}}. \tag{33}$$

This isomorphism is explicitly realized by fixing orthonormal bases $\boldsymbol{h}^{(i)}$, and $\boldsymbol{b}^{(i)}$ for $H^i_{\mathfrak{g}}$ and $B^i_{\mathfrak{g}}$, respectively. Thus, if we denote by $\nu_i \equiv \wedge_q^{\dim(H)^i} h_q^{(i)}$ the corresponding volume form in $H^i_{\mathfrak{g}}$ (lifted to $C^i_{\mathfrak{g}}$), we can write

$$\begin{aligned}&[\wedge_q^{\dim(B)^i} b_q^{(i)}] \wedge [\wedge_q^{\dim(B)^{i+1}} \mathrm{d}b_q^{(i+1)}] \wedge [\wedge_q^{\dim(H)^i} h_q^{(i)}] \\ &\quad = t_i(\nu_i)[\wedge^{\dim(C)^i} \hat{\sigma}^i_j \otimes X_k]\end{aligned} \tag{34}$$

for some scalar $t_i(\nu_i) \neq 0$.

With these remarks out of the way, and setting, for notational convenience, $\mu_i \equiv \wedge^{\dim(C)^i} \hat{\sigma}^i_j \otimes X_k$, we can define the Reidemeister torsion associated with the cochain complex $C^*_{\mathfrak{g}}$ according to the following definition.

Definition 6. For a given choice of volume elements ν_i in $H^*_{\mathfrak{g}}$, the torsion of the cochain complex $C^*_{\mathfrak{g}}$ is the volume element

$$\Delta^{\mathfrak{g}}(M;\mu,\nu) \equiv t(\mathfrak{g}_\theta) = \prod_i [t_i(\nu_i)]^{(-1)^i} \in \text{detline}|C^*(M,\mathfrak{g}_\theta)|. \tag{35}$$

Notice that we have selected a particular definition out of many naturally equivalent ones (see [RS] for a more detailed treatment).

As the notation suggests, it is easily checked that $\Delta^{\mathfrak{g}}(M;\mu,\nu)$ is independent of the particular choice of the bases $\{\boldsymbol{b}\}^{(i)}$ for the $B^i_{\mathfrak{g}}$. Moreover, if the complex $C^i_{\mathfrak{g}}$ is acyclic then $\Delta^{\mathfrak{g}}(M;\mu,\nu)$ is also independent from the choice of a volume element in $H^*_{\mathfrak{g}}$ (recall that the cochain complex $C^i_{\mathfrak{g}}$ is said to be acyclic if $H^i_{\mathfrak{g}} = 0$ for all i).

Obviously, we may have worked as well in homology $H^{\mathfrak{g}}_*$, by obtaining $\Delta^{\mathfrak{g}}(M;\mu,\nu)$ as an element of detline$|C_*(M,\mathfrak{g}_\theta)|$ depending now on a choice of volume elements ν^i in the homology groups $H^{\mathfrak{g}}_i$.

It is important to stress that if the complex $C^*(M,\mathfrak{g}_\theta)$ is not acyclic then $\Delta^{\mathfrak{g}}(M;\mu,\nu)$ is not a scalar but a volume element in detline$|H^*(M,\mathfrak{g}_\theta)|$ under the natural identification between this latter line bundle and detline$|C^*(M,\mathfrak{g}_\theta)|$.

The torsion is an interesting combinatorial invariant of a complex, since it is invariant under subdivision of M and it is deeply related to homotopy theory. In particular, given a homotopy equivalence $f:(M_1,\mathcal{N}_1) \to (M_2,\mathcal{N}_2)$ between two cellular complexes, there is a correspondence between the flat bundles over M_1 and the flat bundles over M_2 induced by the isomorphism $\pi_1(M_1) \to \pi_1(M_2)$ and by the representation θ of such groups into the Lie group G. However, the corresponding torsions are not necessarily equal, this being the case if and only if h is (homotopic to) a Piecewise-Linear (PL) equivalence between the complexes in question.

Also notice that if the manifold M underlying the complex is an orientable, even-dimensional manifold *without boundary* and the cochain complex $C_*(M, \mathfrak{g}_\theta)$ is acyclic, then $\Delta^{\mathfrak{g}}(M; \mu, \nu) = 1$ (see e.g., [Ch]). Thus it would seem that calling into play such invariant for counting geodesic ball coverings over a four-dimensional manifold, is useless. However, there are three reasons which show that the role of torsion is not so trivial for our counting purposes. First, we shall deal with the torsions of the geodesic balls which are four-dimensional manifolds with a non-trivial three-dimensional boundary. Moreover, the complexes we need to use are not acyclic, and the behavior of volume elements ν in cohomology will play a basic role. Finally, the fact that we are in dimension four will be imposed only in the final part of our paper, when estimating the dimension of the tangent space to the set of all conjugacy classes of representations of the fundamental group, (see below). In this connection, we wish to stress that *the analysis which follows holds for any n-dimensional manifold $M \in \mathcal{R}(n, r, D, V)$ with $n \geq 2$.*

We now examine the dependence of $\Delta^{\mathfrak{g}}(M; \mu, \nu)$ on the particular representation $\theta : \pi_1(M) \to G$. To this end, let $\mathrm{Hom}(\pi_1(M), G)/G$ denote the set of all conjugacy classes of representations of the fundamental group $\pi_1(M)$ into the Lie group G. Notice that if θ and $F\theta F^{-1}$ are two conjugate representations of $\pi_1(M)$ in G, then through the map $\mathrm{Ad}(F) : \mathfrak{g} \to \mathfrak{g}$ we get a natural isomorphism between the groups $H_i(M, \mathfrak{g})$ and $H_i(M, F\mathfrak{g}F^{-1})$. Thus it follows that the torsion corresponding to the representation θ and the torsion corresponding to the conjugate representation $F\theta F^{-1}$ are equal, and $\Delta^{\mathcal{G}}(M; \mu, \nu)$ is actually well defined on the conjugacy class of representations $[\theta] \in \mathrm{Hom}(\pi_1(M), G)/G$.

When defining the Reidemeister torsion, one of the advantages of using the homology $H_*(M, \mathfrak{g})$ with local coefficients in the bundle $\mathfrak{g}_\theta$ lies in the fact that the corresponding cohomology is strictly related to the structure of the representation space $\mathrm{Hom}(\pi_1(M), G)/G$. This point is quite important, since we are interested in understanding the dependence of $\Delta^{\mathfrak{g}}(M; \mu, \nu)$ when deforming the particular representation $\theta : \pi_1(M) \to G$ through a differentiable one-parameter family of representations θ_t with $\theta_0 = \theta$ which are not tangent to the G-orbit of $\theta \in \mathrm{Hom}(\pi_1(M), G)$.

To this end, let us rewrite, for t near 0, the given one-parameter family of representations θ_t as [Go,Wa]

$$\theta_t = \exp[tu(a) + \mathrm{O}(t^2)]\theta(a), \tag{36}$$

where $a \in \pi_1(M)$, and where $u : \pi_1(M) \to \mathfrak{g}$. In particular, given a and b in $\pi_1(M)$, if we differentiate the homomorphism condition $\theta_t(ab) = \theta_t(a)\theta_t(b)$, we get that u actually is a one-cocycle of $\pi_1(M)$ with coefficients in the $\pi_1(M)$-module $\mathfrak{g}_\theta$, viz.,

$$u(ab) = u(a) + [\mathrm{Ad}(\theta(a))]u(b). \tag{37}$$

Moreover, any u verifying the above cocycle condition leads to a map $\theta_t : \pi_1(M) \to G$ which, to first order in t, satisfies the homomorphism condition. This remark implies that the (Zariski) tangent space to $\mathrm{Hom}(\pi_1(M), G)$ at θ, can be identified with $Z^1(M, \mathfrak{g}_\theta)$.

In a similar way, it can be shown that the tangent space to the Ad-orbit through θ is $B^1(M, \mathfrak{g}_\theta)$. Thus, the (Zariski) tangent space to $\mathrm{Hom}(\pi_1(M), G)/G$ corresponding to the conjugacy class of representations $[\theta]$ is $H^1(M, \mathfrak{g}_\theta)$. And, as it is usual in deformation theory, this is the formal tangent space to the representation space.

It must be emphasized that, in general, there are obstructions [Go] that do not allow the identification between the Zariski tangent space with the actual tangent space to $\mathrm{Hom}(\pi_1(M), G)/G$. Typically we have troubles in correspondence to reducible representations. Since the tangent space to the isotropy group of the representation θ, is $H^0(M, \mathfrak{g}_\theta)$, it follows that $H^0(M, \mathfrak{g}_\theta) \neq 0$ precisely when there are reducible representations. Further obstruction to identifying $H^1(M, \mathfrak{g}_\theta)$ to the actual tangent space are in $H^2(M, \mathfrak{g}_\theta)$. In deformation theory it is well known that this space is to contain the obstructions to extend a formal deformation to a finite deformation (i.e., in a language more familiar to relativists, $H^2(M, \mathfrak{g}_\theta)$ is associated to a linearization instability around the given representation in $\mathrm{Hom}(\pi_1(M), G)/G$). The triviality of this space at a (conjugacy class of a) representation θ is a necessary condition for θ to be a regular point of the representation space $\mathrm{Hom}(\pi_1(M), G)/G$, and for identifying $H^1(M, \mathfrak{g}_\theta)$ with $T_\theta[\mathrm{Hom}(\pi_1(M), G)/G]$.

We shall be ignoring the singularities produced by reducible representations by restricting our considerations to the set of irreducible representations $\mathcal{S} \in \mathrm{Hom}(\pi_1(M), G)/G$, yet, in general we do not assume that $H^2(M, \mathfrak{g}_\theta) = 0$. Not considering reducible representations is certainly not topologically justified in general, but is not yet clear how to circumvent the difficulties associated with them. Moreover, the results we obtain are well-defined in considerable generality and do not seem to suffer too much by such restrictions.

Recall that $\mathrm{Hom}\,(\pi_1(X), G)^{\mathrm{irr}}$ is a smooth analytic submanifold of G^m for some m. This provides $\mathcal{S}$ with an analytic structure, possibly outside some singular points. Let $\mathcal{S}_0$ denote the smooth locus of $\mathcal{S}$, and let d be its dimension.

To be definite, we set $G = U(n)$. If we assume M to be oriented, the space $\mathcal{S}_0$, regarded as the space of gauge equivalence classes of flat connections ∇ on $\mathfrak{g}_\theta$, sits inside both $\mathcal{M}_+$ and $\mathcal{M}_-$, these two spaces being the moduli spaces of self-dual and antiself-dual irreducible instantons on M.

Since

$$\dim \mathcal{M}_- = \dim G(b_1 - b_+ - 1)\,, \qquad \dim \mathcal{M}_+ = \dim G(b_1 - b_- - 1)\,, \tag{38}$$

we get the inequalities

$$d \leq \dim G(b_1 - b_+ - 1), \qquad d \leq \dim G(b_1 - b_- - 1)\,; \tag{39}$$

by summing the two inequalities we get

$$d \leq -\tfrac{1}{2}\dim G\chi(M)\,. \tag{40}$$

We stress that d is the dimension of the representation variety in the neighborhood of smooth points. In general d is different from the Zariski dimension as computed by the cohomology $H^*(M, \mathfrak{g}_\theta)$. Let us define $b(k) = \dim H^k(M, \mathfrak{g}_\theta)$. Then $b(0) = b(4) = 0$ by irreducibility (and due to Poincaré duality), while $b(1) = b(3)$. Recall that the space $H^1(M, \mathfrak{g}_\theta)$ can be thought of as the Zariski tangent space to $\mathcal{S}$ at $[\theta]$; let us denote $d_Z(\theta) = b(1)$. So $d_Z = d$ at a smooth point, while $d_Z \geq d$ in general (see [GM]). Indeed a non-vanishing $H^2(M, \mathfrak{g}_\theta)$

may represent an obstruction to the identification of the Zariski tangent space to the smooth tangent space at the point $[\theta]$.

The Zariski dimension d_Z may be computed by using the Atiyah–Singer index theorem. Let $\Omega^p(\mathfrak{g}_\theta)$ denote the space of all $\mathfrak{g}_\theta$-valued exterior p-forms on M. We may consider the elliptic complex

$$0 \to \Omega^0(\mathfrak{g}_\theta) \to \Omega^1(\mathfrak{g}_\theta) \to \Omega^2(\mathfrak{g}_\theta) \to \Omega^3(\mathfrak{g}_\theta) \to \Omega^4(\mathfrak{g}_\theta) \to 0, \tag{41}$$

whose cohomology is isomorphic to $H^*(M, \mathfrak{g}_\theta)$. The index of the complex (41), ind $= 2d_Z(\theta) - b(2)$, may be computed explicitly getting ind $= -\dim G\chi(M)$, so that

$$d_Z(\theta) = -\tfrac{1}{2}\dim G\chi(M) + \tfrac{1}{2}b(2), \tag{42}$$

where the θ-dependence is implicit in the twisted Betti number $b(2) = \dim H^2(M, \mathfrak{g}_\theta)$.

4. Counting minimal coverings

It is known that, under suitable hypotheses, the Reidemeister torsion can count closed (periodic) orbits of a flow on a (hyperbolic) riemannian manifold [RS]. Our purpose is to show that it can also count inequivalent geodesic ball coverings.

This result is basically a consequence of a cardinality law satisfied by the torsion.

Let A and B denote subcomplexes of the manifold M (as usual thought of as a cellular or a simplicial complex) with $M = A \cup B$, and let us consider a representation $\theta \in \mathrm{Hom}(\pi_1(M), G)$. Let us assume that, for every i, volume elements $\mu_i(A)$, $\mu_i(B)$, and $\nu_i(A)$, $\nu_i(B)$ are chosen for the cochain complexes $C^i_{\mathfrak{g}}(A)$, $C^i_{\mathfrak{g}}(B)$, and the corresponding cohomology groups $H^i_{\mathfrak{g}}(A)$, $H^i_{\mathfrak{g}}(B)$, respectively. Let us further assume that such volume elements determine the volume elements on $C^i_{\mathfrak{g}}(M)$ and $H^i_{\mathfrak{g}}(M)$. Corresponding to this choice of volumes let us denote by $\Delta^{\mathfrak{g}}(M|A)$, $\Delta^{\mathfrak{g}}(M|B)$, and $\Delta^{\mathfrak{g}}(M|A \cap B)$ the Reidemeister–Franz torsions associated with the subcomplexes A, B and $A \cap B$ respectively. Then

$$\Delta^{\mathfrak{g}}(H_{A,B})\Delta^{\mathfrak{g}}(M|A \cup B)\Delta^{\mathfrak{g}}(M|A \cap B) = \Delta^{\mathfrak{g}}(M|A)\Delta^{\mathfrak{g}}(M|B), \tag{43}$$

where $H_{A,B}$ is the long exact cohomology sequence associated with the short exact sequence generated by the complexes $C^*_{\mathfrak{g}}(A \cup B)$, $C^*_{\mathfrak{g}}(A) \oplus C^*_{\mathfrak{g}}(B)$, and $C^*_{\mathfrak{g}}(A \cap B)$ (the correction term $\Delta^{\mathfrak{g}}(H_{A,B})$, associated with the twisted cohomology groups of the above three cochain complexes, disappears when the representation is acyclic).

In order to exploit this cardinality law, let us consider all possible minimal geodesic ϵ-ball coverings of a manifold of bounded geometry $M \in \mathcal{R}(n, r, D, V)$ with a given filling function $\lambda = N^{(0)}_\epsilon(M)$. Given a sufficiently small $\epsilon > 0$ (in particular, smaller than the ϵ_0 provided by Zhu's theorem) and given a representation $\theta : \pi_1(M) \to G$, and still denoting by θ its restrictions to representations of the various $\pi_1(B_\epsilon(p_i))$, we can consider the cohomologies with local coefficients in $\mathfrak{g}_\theta$, $H^*_{\mathfrak{g}}(B_\epsilon(p_i))$ for $i = 1, \ldots, \lambda$. We can use them as labels to distribute over the unlabeled balls $\{B_\epsilon(p_i)\}$. Obviously, the *coordinate* labeling of the balls arising from the centers $\{p_i\}$ are to be factored out to the effect that the balls $\{B_\epsilon(p_i)\}$ are considered as a collection of $\lambda = N_\epsilon^{(0)}(M)$ empty boxes over which distribute the colors $H_\theta^*(B_\epsilon(p_i))$. This must be done according to the constraint expressed by the Mayer–Vietoris sequence, associated with the intersection pattern of the covering, so as to reproduce $H_\mathfrak{g}^*(\cup B_\epsilon(p_i)) \simeq H_\mathfrak{g}^*(M)$.

We formalize these remarks as follows.

Let us assume that $M \in \mathcal{R}(n, r, D, V)$ has diameter diam(M), (diam$(M) \leq D$) and Ricci curvature Ric$(M) \geq r$. Let us consider the generic ball $B_\epsilon(p_i) \subset M$ as a riemannian manifold with boundary, with metric tensor $g_\epsilon(p_i)$. According to the coarse-grained point of view, we can assume that such geodesic ball is obtained, by a rescaling, from a corresponding ball $\tilde{B}_a(\mathrm{diam}(M))$ of radius diam(M) in a space form $\tilde{M}_a^r$ of constant curvature r. Notice that different balls, say $B_\epsilon(p_i)$ and $B_\epsilon(p_k)$, with $i \neq k$, may arise from different space forms, resulting from different quotients of the simply connected space of dimension n and constant curvature r. Thus, for ϵ sufficiently small, all balls are locally isometric, but possibly with different underlying topologies.

In particular, as far as the metric properties are concerned, we assume that

$$g_\epsilon(p_i) = \frac{2\epsilon^2}{\mathrm{diam}(M)^2}\tilde{g}_r \tag{44}$$

for every ball $B_\epsilon(p_i)$ with ϵ sufficiently small, and where $\tilde{g}_r$ denotes the constant curvature metric on the space form $\tilde{M}_a^r$. In terms of the filling function $N_\epsilon^{(0)}(M) = \lambda(\epsilon)$, it is straightforward to check that (44) can be equivalently rewritten as

$$g_\epsilon(p_i) = \rho(M)^{-2/n}\lambda(\epsilon)^{-2/n}\tilde{g}_r, \tag{45}$$

where $\rho(M)$ is a suitable function, depending on the parameters n, r, D, V, and on the actual geometry of the manifold M.

For later use, it is also convenient to introduce the deformation parameter

$$t(\epsilon) \equiv \ln[\rho(M)^{-2/n}\lambda(\epsilon)^{-2/n}] \tag{46}$$

so that we can describe the rescaling (44) as obtained through a smooth one-parameter family of conformal deformation

$$g_t(p_i) = \mathrm{e}^{t(\epsilon)}\tilde{g}_r \tag{47}$$

interpolating between $\tilde{g}_r$ (corresponding to $t = 0$) and the actual $g_\epsilon(p_i)$.

As far as topology is concerned, since the ball $B_\epsilon(p_i)$ comes from the rescaling of the reference ball $\tilde{B}_a(\mathrm{diam}(M))$ in the space form $\tilde{M}_a^r$, we can write $H_\mathfrak{g}^*(B_\epsilon(p_i)) \simeq H_\mathfrak{g}^*(\tilde{B}_a(\mathrm{diam}(M)))$. Thus, with the balls $B_\epsilon(p_1), B_\epsilon(p_2), \ldots, B_\epsilon(p_\lambda)$ we can associate the cohomology groups $H_\mathfrak{g}^q(B_{\bar{\epsilon}}(p_i)) = H_\mathfrak{g}^*(\tilde{B}_{a(i)}(\mathrm{diam}(M)))$, where $a(i)$ labels the possibly inequivalent space forms $\tilde{M}_a^r$ after which the balls $\{B_\epsilon(p_i)\}$ are modeled. Notice that in general $a(i) = a(k)$ for some pair $i \neq k$ since the balls $B_\epsilon(p_i)$ and $B_\epsilon(p_k)$ may be modeled after the same space form $\tilde{M}_a^r$.

4.1. Scaling of torsions

At this stage, there is an important point we wish to stress, namely that even if the twisted cohomology of each ball is not affected by the dilation of the ball, the corresponding volume elements in cohomology do change. In particular, let

$$\bar{\mu}_q(i) \equiv \mu_q(\tilde{B}_{a(i)}(\mathrm{diam}(M))) \tag{48}$$

and

$$\bar{\nu}_q(i) \equiv \nu_q(\tilde{B}_{a(i)}(\mathrm{diam}(M))), \tag{49}$$

respectively, denote chosen (reference) volume elements for the cochain complex $C^q_{\mathfrak{g}}(\tilde{B}_{a(i)}(\mathrm{diam}(M)))$, and for the cohomology group $H^q_{\mathfrak{g}}(\tilde{B}_{a(i)}(\mathrm{diam}(M)))$ associated with the reference ball $\tilde{B}_{a(i)}(\mathrm{diam}(M))$ corresponding to $B_\epsilon(p_i)$. The effect, on the above reference volumes, of scaling to ϵ the radius of such ball, is described by the following lemma.

Lemma 2. *Let $\lambda = N^{(0)}_\epsilon(M)$ denote the value of the filling function as a function of ϵ, then, as the radius of the reference ball varies from* diam(M) *to its actual value, the volume elements $\bar{\mu}_q(i)$ and $\bar{\nu}_q(i)$ scale, as a function of λ, according to*

$$\frac{\nu_q(i)}{\mu_q(i)}(\lambda(\epsilon)) = \frac{\bar{\nu}_q(i)}{\bar{\mu}_q(i)}\lambda(\epsilon)^{-(2/n)(q-n/2)b(q)}, \tag{50}$$

where the Betti number $b(q)$ (in real singular homology) is the dimension of the cohomology group $H^q_{\mathfrak{g}}(\tilde{B}_{a(i)}(\mathrm{diam}(M)))$, and where $\mu_q(i)$ and $\nu_q(i)$ respectively denote the volume elements for the cochain complex $C^q_{\mathfrak{g}}(B_\epsilon(p_i))$, and for the cohomology group $H^q_{\mathfrak{g}}(B_\epsilon(p_i))$ associated with the given ball $B_\epsilon(p_i)$.

This result provides a basic *anomalous scaling* relation satisfied by the ratio of the volume elements $\bar{\mu}_q(i)$ and $\bar{\nu}_q(i)$ as the radius, diam(M), of the generic reference geodesic ball is shrinked to ϵ.

To prove this lemma we first evaluate $(\mathrm{d}/\mathrm{d}t)(\nu_q(i)/\mu_q(i))$, corresponding to the deformation (47), and then integrate (in t) the resulting expression between 0 and t. This can be done by an obvious extension of a construction discussed in the paper by Ray and Singer, [RS], whereby we proceed by considering the ratio of volume elements $(\nu_q(i)/\mu_q(i))$ as generated by a proper choice of a base in $C^q_{\mathfrak{g}}(B_\epsilon(p_i))$.

Proof. Let us denote by $\mathcal{D}^k_{\mathfrak{g}}(B_\epsilon(p_i))$ the space of C^∞-differential forms on $B_\epsilon(p_i)$ with values in the flat bundle $\mathfrak{g}_\theta|B_\epsilon(p_i)$, and which satisfy relative boundary conditions at each point of the boundary $\partial B_\epsilon(p_i)$ (for a definition of such boundary conditions see Ray–Singer, *ibidem* p.162). Corresponding quantities are similarly defined also for the reference ball $\tilde{B}_{a(i)}(\mathrm{diam}(M))$.

Let $\boldsymbol{h}^q(t) \in \mathcal{H}^q$ be an orthonormal base of harmonic q-forms (with coefficients in $\mathfrak{g}_\theta$) in the space $\mathcal{H}^q \subset \mathcal{D}^k_{\mathfrak{g}}(B_\epsilon(p_i))$ of harmonic forms associated with the metric g_t. Let $A^q : \mathcal{H}^q \to C^q_{\mathfrak{g}}(B_\epsilon(p_i))$ denote the twisted deRham map

$$A^q \boldsymbol{h}(\xi \otimes \sigma) = \int_\sigma \operatorname{tr}(\xi, \boldsymbol{h}), \tag{51}$$

where σ is a q-cell in $B_\epsilon(p_i)$, $\xi \in \mathfrak{g}_\theta$, and $\operatorname{tr}(\cdot,\cdot)$ denotes the inner product in $\mathfrak{g}_\theta$. Since A^q is an injective map of $\mathcal{H}^q$ onto a linear space of cocycles representing $H^q_{\mathfrak{g}}(B_\epsilon(p_i))$, we may use it as a part of a base for $C^q_{\mathfrak{g}}(B_\epsilon(p_i))$. Choose a base $\boldsymbol{b}^q = \{b^q_j\}$ for the space of coboundaries $B^q_{\mathfrak{g}}(B_\epsilon(p_i))$ and for each b^{q+1}_j take an element $\tilde{b}^{q+1}_j$ of $C^q_{\mathfrak{g}}(B_\epsilon(p_i))$ such that $\mathrm{d}\tilde{b}^{q+1}_j = b^{q+1}_j$. Both b^q_j and $\tilde{b}^{q+1}_j$ can be chosen independently of the metric g_t. Thus $(b^q_j, \tilde{b}^{q+1}_j, A^q(h^q_j))$ is a base for $C^q_{\mathfrak{g}}(B_\epsilon(p_i))$ depending from the metric g_t only through the base of harmonic forms h^q_j. Following Ray–Singer, we denote by D^q the matrix providing the transformation from the base $\hat{\sigma}^q_j X_k$ of $C^q_{\mathfrak{g}}(B_\epsilon(p_i))$, generated by the cells of $C^q(B_\epsilon(p_i))$ and the orthonormal base $\mathfrak{g}_\theta$, and the base $(b^q_j, \tilde{b}^{q+1}_j, A^q(h^q_j))$ introduced above. Thus

$$\frac{\nu_q(i)}{\mu_q(i)} = |\det D^q|. \tag{52}$$

The computation of the derivative of the determinant of D^q is carried out in Ray–Singer, [RS], where it is explicitly applied to the discussion of the behavior of the Reidemeister torsion as the metric varies (see their Theorem 7.6). Explicitly, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln \frac{\nu_q(i)}{\mu_q(i)} = \sum_j^{b(q)} \left(h^q_j, \frac{\mathrm{d}}{\mathrm{d}t} h^q_j \right)_{L^2}, \tag{53}$$

where $b(q)$ is the Betti number (in real singular homology) of $H^q_{\mathfrak{g}}(B_\epsilon(p_i))$, and $(\cdot,\cdot)_{L^2}$ denotes the global L^2-inner product in the space of $\mathfrak{g}_\theta$-twisted harmonic q-forms $\mathcal{H}^q$, namely, for any two such forms $\boldsymbol{f}$ and $\boldsymbol{g}$,

$$(\boldsymbol{f}, \boldsymbol{g})_{L^2} = \int_{B_\epsilon(p_i)} \operatorname{tr}(\boldsymbol{f} \wedge *\boldsymbol{g}), \tag{54}$$

where $*$ denotes the Hodge-duality operator, and tr stands for the inner product in $\mathfrak{g}_\theta$.

The global inner product $(h^q_j, \mathrm{d}h^q_j/\mathrm{d}t)_{L^2}$ is easily evaluated corresponding to the conformal deformation (47). Indeed, we may rewrite $(h^q_j, \mathrm{d}h^q_j/\mathrm{d}t)_{L^2}$ as $(h^q_j, *^{-1}\,\mathrm{d} * h^q_j/\mathrm{d}t)_{L^2}$ (see e.g., Proposition 6.4 of Ray–Singer [RS]). A straightforward computation shows that the derivative of the Hodge map $*_t$, associated to the t-flow of metrics g_t, defined by (47), is provided by

$$\left.\frac{\mathrm{d}*_t}{\mathrm{d}t}\right|_{t=0} \boldsymbol{f} = [q - \tfrac{1}{2}n] * \boldsymbol{f} \tag{55}$$

for any given q-form $\boldsymbol{f}$ with values in the flat bundle $\mathfrak{g}_\theta$.

Thus

$$\left.\left(h^q_j, \frac{\mathrm{d}}{\mathrm{d}t} h^q_j \right)_{L^2}\right|_{t=0} = [q - \tfrac{1}{2}n] \int_{B_{\epsilon/2}(p_i)} \operatorname{tr}(h^q_j \wedge * h^q_j) = [q - \tfrac{1}{2}n], \tag{56}$$

since the basis h_j^q is orthonormal.

Introducing this latter expression in (53) we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln \frac{\nu_q(i)}{\mu_q(i)} = b(q)[q - \tfrac{1}{2}n]. \tag{57}$$

We integrate (57) with the initial condition

$$\frac{\nu_q(i)}{\mu_q(i)}(t=0) = \frac{\bar{\nu}_q(i)}{\bar{\mu}_q(i)}, \tag{58}$$

where $\bar{\nu}_q(i)$ and $\bar{\mu}_q(i)$ respectively refer to the original unscaled measures on the cochain complex $C^q_{\mathfrak{g}}(\tilde{B}_{a(i)}(\mathrm{diam}(M)))$ and on the cohomology group $H^q_{\mathfrak{g}}(\tilde{B}_{a(i)}(\mathrm{diam}(M)))$.

With this initial condition, and if we take into account the explicit expression of t in terms of the filling function λ we get

$$\frac{\nu_q(i)}{\mu_q(i)}(t(\lambda)) = \frac{\bar{\nu}_q(i)}{\bar{\mu}_q(i)}[\rho^{-2/n}\lambda^{-2/n}]^{(q-n/2)b(q)}. \tag{59}$$

Thus, we eventually get

$$\frac{\nu_q(i)}{\mu_q(i)}(\lambda(\epsilon)) = \frac{\bar{\nu}_q(i)}{\bar{\mu}_q(i)}\lambda(\epsilon)^{-(2/n)(q-n/2)b(q)}, \tag{60}$$

where we have traded the term $[\rho^{-2/n}]^{(q-n/2)b(q)}$ for a redefinition of the given original unscaled measures $\bar{\nu}_q(i)$ and $\bar{\mu}_q(i)$. This completes the proof of Lemma 2. □

Corresponding to this rescaling of volume elements, we can evaluate the relation between Reidemeister torsion, for the generic geodesic ball $B_\epsilon(p_i)$, as expressed in terms of the scaled $\nu_q(i)$, $\mu_q(i)$ and unscaled measures $\bar{\nu}_q(i)$, $\bar{\mu}_q(i)$. A straightforward computation yields

$$\Delta^{\mathfrak{g}}(B_\epsilon(p_i); \mu(i), \nu(i)) = \Delta^{\mathfrak{g}}(B_\epsilon(p_i); \bar{\mu}(i), \bar{\nu}(i))\lambda^{-(2/n)\sum_q(-1)^q(q-n/2)b(q)} \tag{61}$$

Notice that the exponent of λ, viz., $\sum_q(-1)^q(1-2q/n)b(q)$ vanishes, by Poincaré duality, if the ball is compact and without boundary (in particular it vanishes when $\epsilon \to \mathrm{diam}(M)$, namely when the ball $B_\epsilon(p_i)$ is expanded so as to cover the whole manifold M). In this sense, it is a measure of the presence of the boundary. If we set

$$\alpha(i) \equiv [\dim(G)]^{-1}\sum_q(-1)^q q b(q) \tag{62}$$

and recall that $\sum_q(-1)^q b(q) = \dim(G)\chi(i)$, where $\chi(i) \equiv \chi(B_\epsilon(p_i))$ is the Euler–Poincaré characteristic of the given ball $B_\epsilon(p_i)$, then we can rewrite (61) as

$$\Delta^{\mathfrak{g}}(B_\epsilon(p_i); \mu(i), \nu(i)) = \Delta^{\mathfrak{g}}(B_\epsilon(p_i); \bar{\mu}(i), \bar{\nu}(i))\lambda^{\dim(G)(2/n)[(n/2)\chi(i)-\alpha(i)]}. \tag{63}$$

4.2. Distinct coverings in a given representation of $\pi_1(M)$

With these preliminary remarks out of the way, our strategy is to construct, out of the sequence of λ balls $\{B_\epsilon(p_i)\}$, each endowed with the metric $g_t(p_i)$, all possible geodesic ball coverings providing and ϵ-Hausdorff approximation to the original M. In order to do so, we need to consider explicitly the generalized Meyer–Vietoris sequence associated with the covering $\{B_\epsilon(p_i)\}$ (see [BT] for details). To simplify the notation, we shall denote by $B(i)$, with $i = 1, \ldots, \lambda$, the generic open ball $B_\epsilon(p_i)$. Similarly, we denote the pairwise intersections $B(i) \cap B(j)$ by $B(i, j)$, triple intersections $B(i) \cap B(j) \cap B(k)$ by $B(i, j, k)$, and so on. Recall that for a manifold of bounded geometry, the number of mutually intersecting balls is bounded above by a constant d, depending on the parameters n, r, D characterizing $\mathcal{R}(n, r, D, V)$, but otherwise independent from ϵ. Thus, independently from ϵ, the largest cluster of mutually intersecting balls which can occur for any $M \in \mathcal{R}(n, r, D, V)$ is $B(i_1, i_2, \ldots, i_d)$.

As usual [BT], we denote by ∂_η the inclusion map which ignores the i_η open ball $B(i_\eta)$ in the generic cluster $B(i_1, \ldots, i_\eta, \ldots)$. For instance

$$\partial_i : \coprod_i B(i, j, k) \to B(j, k). \tag{64}$$

By considering the cochain complexes $C^*_{\mathfrak{g}}(B(i, j, \ldots))$ associated with the intersections $B(i, j, \ldots)$, one can consider the restriction map δ_η defined by the image of the cocycles under the pullback map induced by the inclusion ∂_η. For instance, corresponding to the above inclusion we get

$$\delta_i : C^*_{\mathfrak{g}}(B(j, k)) \to \prod_i C^*_{\mathfrak{g}}(B(i, j, k)). \tag{65}$$

Thus, associated with any given minimal geodesic ball covering $\{B(i)\}$, there is a sequence of inclusions relating the intersections $B(i, j, k, \ldots)$ with the packing $\coprod_i B(i)$

$$\cdots \to \coprod_{i<j<k} B(i, j, k) \to \coprod_{i<j} B(i, j) \to \coprod_i B(i) \to M \tag{66}$$

and a corresponding sequence of restrictions

$$C^*_{\mathfrak{g}}(M) \to \prod_i C^*_{\mathfrak{g}}(B(i)) \to \prod_{i<j} C^*_{\mathfrak{g}}(B(i, j)) \to \prod_{i<j<k} C^*_{\mathfrak{g}}(B(i, j, k)) \to \cdots \tag{67}$$

If in this latter sequence we replace the restriction maps with the corresponding difference operator $\delta : \prod C^*_{\mathfrak{g}}(B(i_1, \ldots, i_\eta)) \to \prod C^*_{\mathfrak{g}}(B(i_1, \ldots, i_\eta, i_\gamma))$ defined by the alternating difference $\delta_1 - \delta_2 + \cdots (+/-)\delta_\eta -/+ \delta_\gamma$, then we get the generalized Mayer–Vietoris exact sequence

$$0 \to C^*_{\mathfrak{g}}(M) \to \prod_i C^*_{\mathfrak{g}}(B(i)) \to \prod_{i<j} C^*_{\mathfrak{g}}(B(i, j)) \to \prod_{i<j<k} C^*_{\mathfrak{g}}(B(i, j, k)) \to \cdots \tag{68}$$

The sequences (66)–(68) intermingle the combinatorics of the geodesic ball packings and of the corresponding coverings with the topology of the underlying manifold M.

The function that associates with a manifold of bounded geometry the number of distinct geodesic ball packings extend continuously (in the Gromov–Hausdorff topology) through the Mayer–Vietoris sequence (68). Thus, our strategy will be to enumerate all possible $\frac{1}{2}\epsilon$-geodesic ball packings, modulo a permutation of their centers $\{p_i\}$, and then extend by continuity the resulting counting function to the corresponding coverings.

In order to view a manifold of given fundamental group $\pi_1(M)$ as generated by packing and gluing metric balls we must choose base-points and arcs connecting these points. Only in this way we will be able to consider curves in the balls either as elements of the fundamental groups of the balls themselves or as elements of the fundamental group of the manifold M. So we choose as basepoints in the balls their respective λ centers $p_1, p_2, \ldots, p_\lambda$. One of these centers (say p_1) is then chosen as a basepoint in M. Next we need to choose arcs L_{ij} connecting the points p_i and p_j. This amounts in giving a *reference* intersection pattern for the geodesic ball coverings, namely a *reference one-skeleton* $\Gamma_\epsilon^{(1)}(M; \text{ref})$. If L_i is a path in M, corresponding to a path in the graph $\Gamma_\epsilon^{(1)}(M; \text{ref})$, connecting p_1 with p_i, and C_i is a curve in the ball $B(i)$, then $\hat{C}_i \equiv L_i^{-1} * C_i * L_i^{-1}$ is an equivalence class in $\pi_1(M)$. In this connection, it is particularly helpful that isomorphic (in the combinatorial sense) one-skeleton graphs correspond to manifolds with a same homotopy type.

The remarks above imply that in order to enumerate all possible coverings, we *need* to start by giving a *reference* covering Cov_{ref}

$$\cdots \to \coprod_{i<j<k} B(i,j,k) \to \coprod_{i<j} B(i,j) \to \coprod_{i} B(i) \to M \tag{69}$$

specifying the homotopy type of the manifolds M in $\mathcal{R}(n, r, D, V)$ we are interested in. We wish to stress that this reference covering is common to many topologically distinct manifolds, for we are not specifying a priori the topology of each ball. Recall that according to Gromov's coarse grained point of view, two manifolds M_1 and M_2 in $\mathcal{R}(n, r, D, V)$, having the same ϵ-geodesic ball covering, define an ϵ-Hausdorff approximation of a same manifold, without necessarily being homeomorphic to each other. For ϵ small enough, such approximating manifolds only share the homotopy type, (and hence have isomorphic fundamental groups). Thus the reference covering Cov_{ref}, (69), may be considered as a bookeeping device for fixing the homotopy type (and in particular the fundamental group) of the class of manifolds we are intersted in.

From a combinatorial point of view, Cov_{ref} labels the intersection pattern of centers $\{p_i\}$ of the balls in a given order. The strategy is to determine the number of different ways of associating with such centers the actual balls $\{\tilde{B}_a, H_{\mathfrak{g}}^*(\tilde{B}_a)\}$ after which the geodesic balls are modeled, i.e., we have to fill the *reference* balls $B(i)$ with some topology. Any two such correspondence between centers and model balls are considered equivalent if they can be obtained one from the other through the action of the symmetric group acting on the centers. In this way we avoid to count as distinct the relabelings of the centers of a same pattern of model balls. We prove that in this way, we can obtain all possible coverings.

Let Perm denote the group of permutations of the collection of balls $\{B(i)\} \in \text{Cov}_{\text{ref}}$, namely the symmetric group S_λ acting on the λ centers $\{p_1, \ldots, p_\lambda\}$. Also let $\{C_{\mathfrak{g}}^*(a)\}$, with $a = 1, 2, \ldots, |C_{\mathfrak{g}}^*|$, denote the set of possible cochain groups for the model balls $\{\tilde{B}_a\}$, where $|C_{\mathfrak{g}}^*|$ denotes the cardinality of $\{C_{\mathfrak{g}}^*(a)\}$.

We are tacitly assuming that different balls may have the same $C^*_{\mathfrak{g}}(a)$. But actually, in the final result we allow for $|C^*_{\mathfrak{g}}| \to (n+1)\lambda$. As often emphasized, $\{C^*_{\mathfrak{g}}(a)\}$ is the typical set of *colors* for the balls $B(i)$ coming from the model balls $\{\tilde{B}_a\}$ in the space forms $\tilde{M}^r_a$. Similarly, we denote by $\{C^*_{\mathfrak{g}}(a,b)\} \subset \{C^*_{\mathfrak{g}}(a)\}$ the set of possible cochain groups for the pairwise intersections $B(i,j)$; by $\{C^*_{\mathfrak{g}}(a,b,c)\} \subset \{C^*_{\mathfrak{g}}(a)\}$ the possible cochain groups for the triplewise intersections $B(i,j,k)$, etc. All such groups are assumed to be related by a sequence of restrictions analogous to (67), namely

$$\prod_{a} C^*_{\mathfrak{g}}(a) \to \prod_{a<b} C^*_{\mathfrak{g}}(a,b) \to \prod_{a<b<c} C^*_{\mathfrak{g}}(a,b,c) \to \cdots \tag{70}$$

Remark 2. It is important to stress that even if the balls (and their intersections) are topologically trivial (namely if they are contractible) the labels associated with the $C^*_{\mathfrak{g}}(a)$ are non-trivial. Indeed, for a contractible ball we get

$$H^0_{\mathfrak{g}}(B(i)) \simeq \mathfrak{g}_{\theta_i}, \tag{71}$$

while the remaining twisted cohomology groups all vanish. Thus in this case, the label is provided by local flat bundles over $B(i)$ associated with the representation θ. Since there is no canonical isomorphism between these flat bundles over the balls $B(i)$, we have to assume that the labels $C^*_{\mathfrak{g}}(a)$ are distinct.

Let us consider the set of all functions, $f \equiv (f_{(i)}, f_{(ij)}, f_{(ij\cdots)}, \ldots)$, compatible with the morphisms of the two complexes (66) and (70), where

$$f_{(i_1\cdots i_p)} : \{B(i_1, \ldots, i_p)\} \to \{C^*_{\mathfrak{g}}(a_1, \ldots, a_p)\} \tag{72}$$

is the function which associates with the generic mutual intersection of balls $B(i, j, \ldots)$ the corresponding cochain group $C^*_{\mathfrak{g}}(B(i, j, \ldots)) = C^*_{\mathfrak{g}}(a_1, a_2, \ldots)$ out of the possible ones $\{C^*_{\mathfrak{g}}(a, b, \ldots)\}$.

Let $\sigma \in$ Perm a permutation acting on the balls $\{B(i)\}$. Any such σ can be made to act on the set of function f, by defining

$$(\sigma^* f)(B(i_1, \ldots, i_k)) = f(\sigma B(i_1, \ldots, i_k)) \tag{73}$$

for any $1 \le k \le d$, and where

$$\sigma B(i_1, \ldots, i_k) \equiv \sigma B(i_1) \cap \cdots \cap \sigma B(i_k). \tag{74}$$

Thus, the equivalence class of configurations f under the action of Perm is well defined; it is the *Combinatorial Pattern* of the geodesic ball covering $f(\mathrm{Cov}_{\mathrm{ref}})$ in the representation $[\theta]$.

Notice that if we assume that the reference covering $\mathrm{Cov}_{\mathrm{ref}}$ is explicitly realized on a given manifold M, then the orbit of the map

$$f^{(\mathrm{ref})}_{(i_1\cdots i_p)} : \{B(i_1, \ldots, i_p)\} \to \{C^*_{\mathfrak{g}}(M; i_1, \ldots, i_p)\}, \tag{75}$$

which allocates the balls $\{B(i)\}$ of the reference covering on their centers, corresponds to the given reference covering $\mathrm{Cov}_{\mathrm{ref}}$ and all isomorphic coverings that can be obtained from the reference covering by relabeling the centers of the balls. Not all possible maps $f_{(i_1 \cdots i_p)}$ belong to the orbit of $f^{(\mathrm{ref})}_{(i_1 \cdots i_p)}$ and in general we can prove the following theorem.

Theorem 4. *In a given conjugacy class of (irreducible) representations* $[\theta] \in \mathrm{Hom}(\pi_1(M), G)/G$, *and given a set of possible colors* (70), *any two minimal geodesic* ϵ*-ball coverings* (66) *are distinct if and only if they correspond to distinct orbits of the permutation group* Perm *acting on the set of all functions* $f \equiv (f_{(i)}, f_{(ij)}, f_{(ij\cdots)}, \ldots)$.

Proof. Let $M \in \mathcal{R}(n, r, D, V)$ be a given manifold Let Cov_1 and Cov_2 be ϵ-geodesic ball coverings of M having the same number of balls. They are *isomorphic* if there is an injective mapping h of the balls of Cov_1 onto those of Cov_2 which satisfies the following condition.

(i) Any two distinct balls B_α and B_β of Cov_1 mutually intersect each other if and only if their images $h(B_\alpha)$ and $h(B_\beta)$ mutually intersects each other in Cov_2.

This condition is extended to the mutual intersection of any number ($\leq d$), of balls, and can be rephrased in terms of the nerves associated with the coverings Cov_1 and Cov_2, by saying that vertices of $\mathcal{N}(\mathrm{Cov}_1)$ define a k-simplex if and only if their images under h define a k-simplex in $\mathcal{N}(\mathrm{Cov}_2)$, (see Section 2.1).

Let $f^{(1)} \equiv (f_{(i)}, f_{(ij)}, f_{(ij\cdots)}, \ldots)^{(1)}$ and $f^{(2)} \equiv (f_{(i)}, f_{(ij)}, f_{(ij\cdots)}, \ldots)^{(2)}$ be two functions which are in distinct orbits of the symmetric group. Let us assume that they give rise to two isomorphic geodesic ball coverings according to the definition recalled above. Then there is a mapping h of the balls of the covering $f^{(1)}(\mathrm{Cov}_{\mathrm{ref}})$ onto the balls of the covering $f^{(2)}(\mathrm{Cov}_{\mathrm{ref}})$ such that the corresponding nerves are isomorphic. We can use the map h to relabel the vertices of $f^{(2)}(\mathrm{Cov}_{\mathrm{ref}})$. Thus $f^{(2)}$ and $f^{(1)}$ do necessarily belong to the same orbit of the symmetric group, and we get a contradiction. Conversely, let us assume that $f^{(1)} \equiv (f_{(i)}, f_{(ij)}, f_{(ij\cdots)}, \ldots)^{(1)}$ and $f^{(2)} \equiv (f_{(i)}, f_{(ij)}, f_{(ij\cdots)}, \ldots)^{(2)}$ are in the same orbit of the symmetric group. Then the permutation which maps $f^{(1)}$ to $f^{(2)}$ is an injective mapping of the covering defined by $f^{(1)}$ onto the covering defined by $f^{(2)}$, and the two coverings are isomorphic.

Since the functions f must be compatible with the morphisms of the complexes (66) and (70), and the action of the symmetric group extends naturally through (66), there is no need to consider all functions $f_{(i)}$, $f_{(ij)}$, $f_{(ij\cdots)}$ as varying independently. To generate a geodesic ball covering it suffices to assign the set of all functions, $f \equiv \{f_{(i)}\}$,

$$f_{(i)} : \{B(i)\} \to \{C^*_{\mathfrak{g}}(a)\}, \tag{76}$$

which associate with the generic ball $B(i)$ the corresponding cochain group $C^*_{\mathfrak{g}}(B(i)) = C^*_{\mathfrak{g}}(a)$ out of the possible ones $\{C^*_{\mathfrak{g}}(a)\}$. The remaining functions $f_{(ij\cdots)}$ are then determined by the given reference pattern (66). This circumstance simply corresponds to the fact that the assignement of a *geodesic ball packing*, i.e., of $f_{(i)}$, characterizes a corresponding geodesic ball covering (viz., the one obtained by doubling the radius of the balls) and if we estimate the number of distinct geodesic ball packings we can also estimate the number of the corresponding geodesic ball coverings.

Thus we need to count the number of the distinct patterns associated with the orbits of $f_{(i)}$ under the symmetric group. This can be accomplished through the use of Pólya's enumeration theorem [Bo].

4.3. Entropy function in a given representation of $\pi_1(M)$

Let us write the generic permutation $\sigma \in$ Perm as a product of disjoint cyclic permutations acting on the set of balls $\{B(i)\}$. Denote by $j_k(\sigma)$ the number of cyclic permutations (cycles) of σ of length k. Recall that the *cycle sum* of Perm is the polynomial with integer coefficients in the indeterminates $\{t_k\} = t_1, t_2, \ldots, t_\lambda$ given by

$$C(\mathrm{Perm}; t_1, \ldots, t_\lambda) = \sum_{\sigma \in \mathrm{Perm}} \prod_{k=1}^{\lambda} t_k^{j_k(\sigma)}. \tag{77}$$

Since Perm is in our case the symmetric group S_λ acting on λ objects, we get

$$C(S_\lambda; t_1, \ldots, t_\lambda) = \sum \frac{\lambda!}{\prod_{k=1}^{\lambda} k^{j_k} j_k!} t_1^{j_1} t_2^{j_2} \cdots t_\lambda^{j_\lambda}, \tag{78}$$

where the summation is over all partitions $j_1 + 2j_2 + \cdots + \lambda j_\lambda = \lambda$.

In order to apply Pólya's theorem we need to introduce a function $w : \{C^*_{\mathfrak{g}}(a)\} \to E$ where E is an arbitrary commutative ring. Such w is meant to provide the weight of the possible twisted cochain groups $\{C^*_{\mathfrak{g}}(a)\}$. In this way, one can define the weight of a configuration f of such groups over the packing as

$$w(f) = \prod_i w(f(B(i))). \tag{79}$$

Any two configurations that are equivalent under the action of Perm $= S_\lambda$ have the same weight, and the weight of the pattern associated with the S_λ-orbit $\mathcal{O}_h$ of a f is just $w(\mathcal{O}_h) = w(f)$. By summing over all distinct orbits $\mathcal{O}_h$, with $h = 1, \ldots, l$ one gets the pattern sum

$$S = \sum_{h=1}^{l} w(\mathcal{O}_h), \tag{80}$$

where $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_l$ are the distinct patterns of the geodesic ball packings we wish to enumerate.

Pólya's enumeration theorem (see e.g., [Bo]) relates the above pattern sum to the cycle sum, namely

$$|\mathrm{Perm}| S = C(\mathrm{Perm}; s_1, \ldots, s_\lambda), \tag{81}$$

where $|\mathrm{Perm}|$ is the order of the group of permutations, Perm, considered (thus, $|\mathrm{Perm}| = \lambda!$ in our case) and s_k is the kth figure sum

$$s_k = \sum_a (w(C^*_{\mathfrak{g}}(a)))^k, \tag{82}$$

where the sum extends to all cochain complexes in $\{C^*_{\mathfrak{g}}(a)\}$.

Given the generic cochain $C^*_{\mathfrak{g}}(a)$, a natural candidate for the weight $w(C^*_{\mathfrak{g}}(a))$ is its corresponding torsion

$$w(C^*_{\mathfrak{g}}(a, \bar{\mu}, \bar{\nu})) \equiv \Delta^{\mathfrak{g}}(a), \tag{83}$$

where $\Delta^{\mathfrak{g}}(a)$ is the Reidemeister–Franz torsion of the cochain complex $C^*_{\mathfrak{g}}(a)$ evaluated with respect to the unscaled reference volume elements $\bar{\mu}$ and $\bar{\nu}$ induced by those for the cochain complex $C^*_{\mathfrak{g}}(a)$ and the corresponding cohomology $H^*_{\mathfrak{g}}(a)$.

It is preferable to have these weights expressed in terms of the reference volumes $\bar{\mu}$ and $\bar{\nu}$ rather than the ϵ-scaled volume elements μ and ν, otherwise, according to (63), we would get

$$w(C^*_{\mathfrak{g}}(a; \mu, \nu)) = \Delta^{\mathfrak{g}}(a; \bar{\mu}, \bar{\nu})\lambda^{\dim(G)(2/n)[(n/2)\chi(C^*_{\mathfrak{g}}(a))-\alpha(C^*_{\mathfrak{g}}(a))]}, \tag{84}$$

where we have set

$$\alpha(C^*_{\mathfrak{g}}(a)) = [\dim(G)]^{-1}\Big[\sum_q (-1)^q q b(q; a)\Big] \tag{85}$$

with $\chi(C^*_{\mathfrak{g}}(a))$ and $b(q; a)$ respectively denoting the Euler–Poincaré characteristic and the qth Betti number of $C^*_{\mathfrak{g}}(a)$.

Such a choice for the weight enhances the effect of the boundaries of the balls as follows from the presence of the anomalous scaling term

$$\dim(G)(2/n)[\tfrac{1}{2}n\chi(C^*_{\mathfrak{g}}(a)) - \alpha(C^*_{\mathfrak{g}}(a))]. \tag{86}$$

The entropic contribution of these boundaries to the enumeration of packings can be easily seen to be

$$\lambda^{\dim(G)(2/n)[(n/2)\chi(C^*_{\mathfrak{g}}(a))-\alpha(C^*_{\mathfrak{g}}(a))]\lambda} \tag{87}$$

thus, it is of a factorial nature, and as such quite disturbing in controlling the thermodynamic limit of the theory. As stressed, its origin lies in the fact that by using as reference measures the ϵ-scaled μ and ν, we are implicitly providing an intrinsic topological labeling also for the boundaries of the balls (indeed (86) would vanish, by Poincaré duality, if the ball were closed and without boundary). Such boundary terms are not relevant if we are intersted in coverings, and thus the weight $w(C^*_{\mathfrak{g}}(a; \mu, \nu))$ is too detailed for our enumerative purposes. The proper choice is rather $w(C^*_{\mathfrak{g}}(a; \bar{\mu}, \bar{\nu}))$.

The remarks above are an example of the typical strategy inherent in Pólya's theorem. Indeed, it is exactly the proper choice of the weight to be associated with the colors, that allows one to select the details of interest in the patterns we wish to enumerate.

With these remarks out of the way, it can be easily verified that the weight of a configuration f of the cochain complexes $\{C^*_{\mathfrak{g}}(a)\}$ over the packing $\{B(i)\}$ is nothing but the Reidemeister torsion, in the given representation $[\theta]$ and with respect to the product measures $\prod_i \bar{\mu}_i$, $\prod_i \bar{\nu}_i$, of the disjoint union $\coprod_i \{B(i) \xrightarrow{f} C^*_{\mathfrak{g}}(a)\}$ (this is an immediate application of the cardinality law (43) for the torsion).

We can write down the generic kth figure sum (82), and a standard application of Pólya's theorem would provide, at least in line of principle, the required enumeration of the distinct coverings. However, for large values of the filling function λ explicit expressions are extremely difficult to obtain. Even for small values of λ the evaluation of the cycle sum corresponding to the kth figure sums is unwieldy owing to the non-trivial structure of the weight we are using.

Nonetheless, a useful estimate of the number of distinct covering can be easily extracted from Pólya's theorem. This estimate will be sufficient to characterize in a geometrically significant way the rate of growth, with λ, of the number of geodesic ball packings.

According to Pólya's theorem, we get that

$$\sum_{h=1}^{l} \Delta^{\mathfrak{g}}(O_h) = \sum_{\sigma} \frac{1}{J_1(\sigma)! \cdots J_\lambda(\sigma)!} \left(\frac{\sum_a w(C^*_{\mathfrak{g}}(a))}{1}\right)^{J_1(\sigma)} \cdots \left(\frac{\sum_a w(C^*_{\mathfrak{g}}(a))^\lambda}{\lambda}\right)^{J_\lambda(\sigma)}, \tag{88}$$

where the summation is over all partitions $J_1(\sigma) + 2J_2(\sigma) + \cdots + \lambda J_\lambda(\sigma) = \lambda$.

Since we are interested in the large λ behavior of the above expression, it is convenient to rewrite the figure sums in (88) in a slightly different way.

Let $\tilde{w}(C^*_{\mathfrak{g}}(a))$ denote the value of $w(C^*_{\mathfrak{g}}(a))$ corresponding to which the torsion $\Delta^{\mathfrak{g}}(C^*_{\mathfrak{g}}(a))$ attains its maximum over the set of possible colors $\{C^*_{\mathfrak{g}}(a)\}$, viz.,

$$\tilde{w}(C^*_{\mathfrak{g}}) = \max_a \left\{\Delta^{\mathfrak{g}}(C^*_{\mathfrak{g}}(a))\right\}. \tag{89}$$

Thus, we can write

$$s_k = \sum_a w^k(C^*_{\mathfrak{g}}(a;\mu,\nu)) \le |C^*_{\mathfrak{g}}|\tilde{w}^k(C^*_{\mathfrak{g}}(a;\mu,\nu)), \tag{90}$$

where $|C^*_{\mathfrak{g}}|$ denotes the number of inequivalent cochain groups $C^*_{\mathfrak{g}}(a)$ providing the possible set of colors of the balls $B(i)$.

The generating identity determining the cycle sum for the symmetric group is

$$\sum_{j=0}^{\infty} C(t_1, t_2, \ldots, t_j)u^j/j! = \exp\left(ut_1 + u^2 t_2/2 + u^3 t_3/3 + \cdots\right), \tag{91}$$

where u is a generic indeterminate. For notational convenience, let us set

$$\tau \equiv \tilde{w}(C^*_{\mathfrak{g}}(a;\mu,\nu)). \tag{92}$$

If we replace in (91) t_k with the bound (90) for the figure sum s_k, viz.,

$$t_k = |C^*_{\mathfrak{g}}|\tau^k \tag{93}$$

then, in the sense of generating functions, we get

$$\sum_{j=0}^{\infty} C(t_1, t_2, \ldots, t_j) u^j / j! = \exp\left[|C_{\mathfrak{g}}^*| (u\tau + (u\tau)^2/2 + (u\tau)^3/3 + \cdots) \right]$$
$$= (1 - u\tau)^{-|C_{\mathfrak{g}}^*|}, \tag{94}$$

Thus

$$C(t_1, t_2, \ldots, t_\lambda) = \frac{(|C_{\mathfrak{g}}^*| + \lambda - 1)!}{(|C_{\mathfrak{g}}^*| - 1)!} \tilde{w}^\lambda \tag{95}$$

and according to Pólya's enumeration theorem we get that the pattern sum over all distinct orbits of the permutation group, acting on the $\{C_{\mathfrak{g}}^*(a, b, \ldots)\}$ colored covering (66), is bounded by

$$\sum_{h=1}^{l} \Delta^{\mathfrak{g}}(\mathcal{O}_h) \le \frac{(|C_{\mathfrak{g}}^*| + \lambda - 1)!}{\lambda!(|C_{\mathfrak{g}}^*| - 1)!} \tilde{w}^\lambda. \tag{96}$$

Notice that the combinatorial factor in the above expression is exactly the number of λ-combinations with repetition of $|C_{\mathfrak{g}}^*|$ distinct objects.

The color of each ball $B(i)$ has a degeneracy (the possible *shades*) equal to $n+1$, where n is the dimension of the manifold M. Indeed, since each ball $B(i)$ is topologically non-trivial, its cohomology (with local coefficients) $H_{\mathfrak{g}}^*$ is generated by $n+1$, a priori distinct, groups $H_{\mathfrak{g}}^l$, with $l = 0, 1, \ldots, n$. Since there are λ, a priori distinct, balls we shall set in general

$$|C_{\mathfrak{g}}^*| = (n+1)\lambda; \tag{97}$$

with this assumption, and for $\lambda >> 1$ we get, by applying Stirling's formula

$$\left[\frac{(|C_{\mathfrak{g}}^*| + \lambda - 1)!}{\lambda!(|C_{\mathfrak{g}}^*| - 1)!} \right]_{|C_{\mathfrak{g}}^*| = (n+1)\lambda}$$
$$\simeq \frac{1}{\sqrt{2\pi}} \sqrt{\frac{n+2}{n+1}} \left[\frac{(n+2)^{n+2}}{(n+1)^{n+1}} \right]^\lambda \lambda^{-1/2} \left(1 + \mathrm{O}(\lambda^{-3/2}) \right). \tag{98}$$

It follows from the above results that the asymptotics of the counting function, enumerating the distinct geodesic ball packings with a torsion $\Delta^{\mathfrak{g}}(\mathcal{O}_h)$ in a given representation $[\theta]$, and with respect to the product measures $\prod_i \bar{\mu}_i$, $\prod_i \bar{\nu}_i$, can be read off from the bound

$$\sum_{h=1}^{l} \Delta^{\mathfrak{g}}(\mathcal{O}_h) \le \frac{1}{\sqrt{2\pi}} \sqrt{\frac{n+2}{n+1}} \left[\frac{(n+2)^{n+2}}{(n+1)^{n+1}} \tilde{w} \right]^\lambda \lambda^{-1/2} \left(1 + \mathrm{O}(\lambda^{-3/2}) \right). \tag{99}$$

Explicitly, let $B_{\mathrm{pack}}(\Delta^{\mathfrak{g}}; \lambda)$ denote the number of distinct geodesic ball packings with λ balls and with Reidemeister torsion $\Delta^{\mathfrak{g}}$. In terms of $B_{\mathrm{pack}}(\Delta^{\mathfrak{g}}; \lambda)$ we can write

$$\sum_{h=1}^{l} \Delta^{\mathfrak{g}}(\mathcal{O}_h) = \sum_{\Delta} B_{\mathrm{pack}}(\Delta^{\mathfrak{g}}; \lambda) \Delta^{\mathfrak{g}} \tag{100}$$

Since the bound (99) is a fortiori true for each separate term appearing in the sum, we get an estimate of the asymptotics of the number of distinct geodesic ball packings with torsion $\Delta^{\mathfrak{g}}$

$$B_{\text{pack}}(\Delta^{\mathfrak{q}};\lambda) \leq \frac{1}{\sqrt{2\pi}\,\Delta^{\mathfrak{q}}(\mathcal{O}_h)}\sqrt{\frac{n+2}{n+1}}\left[\frac{(n+2)^{n+2}}{(n+1)^{n+1}}\tilde{w}\right]^{\lambda}\lambda^{-1/2}\left(1+\mathrm{O}(\lambda^{-3/2})\right). \tag{101}$$

Rather unexpectedly, this asymptotics is of an exponential nature, whereas one would have guessed that (allowing for repetitions) there would be a factorial number of ways of distributing $(n+1)\lambda$ distinct labels (the cohomologies $H^*_{\mathfrak{q}}$) over λ *empty* balls. This latter is obviously a correct guess but it does not take into account the action of the symmetric group on the coordinate labelings of the centers of the balls. Since we are interested in distinct (under relabelings) packings, we have to factor out this action. And this reduction is responsible of the transition from a factorial to an exponential growth in (101).

Another relevant aspect of (101) lies in its dependence from the Reidemeister torsion. At this stage, this is simply a consequence of the choice we made for the weight in applying Pólya's theorem. And, had we chosen $w(C^*_{\mathfrak{q}}(a)) = 1$ we would have obtained in place of (101) the estimate

$$\frac{1}{\sqrt{2\pi}}\sqrt{\frac{n+2}{n+1}}\left[\frac{(n+2)^{n+2}}{(n+1)^{n+1}}\right]^{\lambda}\lambda^{-1/2}\left(1+\mathrm{O}(\lambda^{-3/2})\right). \tag{102}$$

This gives a bound to all possible ϵ-geodesic ball packings on a manifold of given fundamental group $\pi_1(M)$, which is consistent with the data coming from numerical simulations, and in dimension $n = 2$ is in remarkable agreement with the known analytical estimates, [BIZ].

The use of the torsion as weight allows for a finer bound, where we can distinguish between different packings (each packing being labeled by the corresponding torsion $\Delta^{\mathfrak{q}}(\mathcal{O}_h;\bar{\mu},\bar{\nu})$). For packings this resolution is not particularly significant, since the torsion of a packing does not have any distinguished topological meaning. However, as we pass from the geodesic ball packing $\coprod_i B(i,\frac{1}{2}\epsilon)$ to the corresponding covering $\cup_i B(i,\epsilon)$, the torsion, now evaluated for the covering, gets identified with the torsion of the underlying manifold. Correspondingly, the bound (101) can be extended by continuity to geodesic ball coverings too. The explicit passage from packings to the corresponding coverings is an elementary applications of Gromov's compactness for the space $\mathcal{R}(n,r,D,V)$, and we get the following proposition.

Proposition 2. *Let $M \in \mathcal{R}(n,r,D,V)$ denote a manifold of bounded geometry with fundamental group $\pi_1(M)$, and let $\theta : \pi_1(M) \to G$ be an irreducible representation of $\pi_1(M)$ into a (semi-simple) Lie group G. For $\epsilon > 0$ sufficiently small, let $\{B_M(p_i,\epsilon)\}$ denote the generic minimal geodesic ball covering of M, whose balls are labeled by the flat bundles $\mathfrak{g}_\theta(B_M(p_i,\epsilon))$ associated with the restrictions of θ to $B_M(p_i,\epsilon)$. If we denote by $N^{(0)}_\epsilon(M) \equiv \lambda$ the filling function of the covering, then, for $\lambda >> 1$, the number, $B_{\mathrm{Cov}}(\Delta^{\mathfrak{q}};\lambda)$, of such distinct geodesic ball coverings is bounded above by*

$$B_{\mathrm{Cov}}(\Delta^{\mathfrak{g}};\lambda) \le \frac{1}{\sqrt{2\pi}\,\Delta^{\mathfrak{g}}(M)}\sqrt{\frac{n+2}{n+1}}\left[\frac{(n+2)^{n+2}}{(n+1)^{n+1}}\tilde{w}\right]^{\lambda}\lambda^{-1/2}\left(1+O(\lambda^{-3/2})\right), \tag{103}$$

where $\Delta^{\mathfrak{g}}(M)$ is the Reidemeister torsion of M in the representation θ.

Proof. From a combinatorial point of view, the injection of the possible $\frac{1}{2}\epsilon$-geodesic ball packings $\coprod_i\{B(i)\}$ into the possible ϵ-geodesic ball coverings, $\cup_i\{B(i)\}$, is a continuous map in the Gromov–Hausdorff topology. It is also consistent with the generalized Meyer–Vietoris sequence (68) associated with the possible coverings. Thus, corresponding to this injection the torsions, $\Delta^{\mathfrak{g}}(\mathcal{O}_h)$, of the possible distinct $\frac{1}{2}\epsilon$-geodesic ball packings, naturally extend to the torsion of the underlying coverings $\cup_i\{B(i)\}$. For a given λ, the bound (99) depends explicitly on the topology of the packing only through these torsions, and the set of possible packings of a manifold of bounded geometry is compact (it is a finite set) in the Gromov–Hausdorff topology. It immediately follows, by Tiezte extension theorem, that (99) has a continuous extension to the counting of all inequivalent geodesic ball coverings of the manifold $M \in \mathcal{R}(n, r, D, V)$ in the given representation $[\theta]$. □

Notice that $\Delta^{\mathfrak{g}}(M)$ plays in (103) the role of a normalization factor. Since there are λ balls $B(i)$ in M, and since the Reidemeister torsion is multiplicative, $\tilde{w}^{\lambda}/\Delta^{\mathfrak{g}}(M)$ would be of the order of 1 if the balls were disjoint (recall that $\tilde{w}$ is the typical torsion of the generic ball). Thus, roughly speaking, the torsion depending factor in (103) is a measure of the *gluing* of the balls of the covering.

The dependence from the representation θ in (103) can be made more explicit. To this end, let us assume that each ball is contractible; then from the cardinality formula for the torsion we get that

$$\tilde{w} = \Delta^{\mathfrak{g}}(B(i)) = \sqrt{\Delta^{\mathfrak{g}}(S^1)}, \tag{104}$$

where $\Delta^{\mathfrak{g}}(S^1)$ is the torsion of the circle S^1 in the given representation θ. Let $A(\theta)$ be the holonomy of a generator of $\pi_1(S^1)$ in the given representation θ. If the matrix $I - A(\theta)$ is invertible, then the flat bundle $\mathfrak{g}_{\theta}$, restricted to the generic ball $B(i)$, is acyclic and

$$\Delta^{\mathfrak{g}}(S^1) = |\det(I - A(\theta))|. \tag{105}$$

Thus (103) can be written explicitly as

$$B_{\mathrm{cov}}(\Delta^{\mathfrak{g}};\lambda) \le \frac{1}{\sqrt{2\pi}\,\Delta^{\mathfrak{g}}(M)}\sqrt{\frac{n+2}{n+1}}\left[\frac{(n+2)^{n+2}}{(n+1)^{n+1}}|\det(I - A(\theta))|\right]^{\lambda}\lambda^{-1/2}\left(1+\mathrm{O}(\lambda^{-3/2})\right). \tag{106}$$

5. Summing over coverings and the volume of the space of riemannian structures

The dependence from the representation θ in (103) and (106) comes from the fact that we are counting distinct coverings on a manifold M endowed with a given flat bundle $\mathfrak{g}_\theta$. We can interpret this in an interesting way by saying that (103) is a functional associating to an equivalence class of representations $[\theta]$ (or which is the same, to each flat bundle or to each gauge equivalence class of flat connections) a statistical weight which in a sense is counting the inequivalent riemannian structures M can carry. As a matter of fact, from a geometric point of view, the term $\tilde{w}^\lambda/\Delta^{\mathfrak{g}}(M)$ is related to a measure density on the representation variety $\mathrm{Hom}(\pi_1(M), G)/G$, and as such it can be used to define an integration on $\mathrm{Hom}(\pi_1(M), G)/G$. The total measure on $\mathrm{Hom}(\pi_1(M), G)/G$ defined by (103) is the the actual entropy function for geodesic ball coverings. By summing this entropy function over all λ we get an expression that can be considered as providing the measure of the set of all riemannian structures of arbitrary volume and of given fundamental group.

In order to elaborate on this point, let us recall that the torsion $\Delta^{\mathfrak{g}}$ is a generalized volume element in $\mathrm{detline}(H^*_{\mathfrak{g}})$. Similarly, we may consider the product $\tilde{w}^\lambda$ as an element of $\mathrm{detline}(H^*_{\mathfrak{g}})$ obtained by pull-back from $\oplus H^*_{\mathfrak{g}}(B(i))$ to $H^*_{\mathfrak{g}}(M)$ according to the Mayer–Vietoris sequence (68). Thus, the ratio $\tilde{w}^\lambda/\Delta^{\mathfrak{g}}$ can be thought of as a density to be integrated over the representation variety.

As recalled in Section 3.2 (see also [JW]) the choice of a representation θ in the equivalence class $[\theta]$ identifies the twisted cohomology group $H^1_{\mathfrak{g}}$ with the Zariski tangent space at $[\theta]$ to the representation variety $\mathrm{Hom}(\pi_1(M), G)/G$. Thus, given a choice of a volume element ν in $H^1_{\mathfrak{g}}$ we may think of $(\tilde{w}^\lambda/\Delta^{\mathfrak{g}})\nu$ as providing a measure on (the dense open set of irreducible representations in) $\mathrm{Hom}(\pi_1(M), G)/G$. This construction is actually very delicate since the representation variety $\mathrm{Hom}(\pi_1(M), G)/G$ is not smooth, and consequently the density bundle may be ill defined. The singularities come from the reducible representations, and given a representation θ, the tangent space to the isotropy group of such θ is $H^0_{\mathfrak{g}}$ (again see Section 3.2 or [JW]). As already stressed we shall be ignoring the singularities of the representation variety in the general setting. One can make an exception for the two-dimensional case, where the structure of $\mathrm{Hom}(\pi_1(M), G)/G$ is better understood.

Given the reference measure ν on $\mathrm{Hom}(\pi_1(M), G)/G$, the associated measure $(\tilde{w}^\lambda/\Delta^{\mathfrak{g}})\nu$ is ill behaved as $\lambda \to \infty$. In order to take care of this problem, we introduce as a damping term the Gibbs factor $\exp[-a\lambda]$ which provides a discretized version of the (exponential of the) volume of the manifold M, with a the (bare) cosmological constant. In this way we have arrived at the natural setting for providing a measure on the space of riemannian structures of given fundamental group induced by the counting function $B_{\mathrm{Cov}}(\Delta^{\mathfrak{g}}; \lambda)$:

$$\begin{aligned} &\mathrm{Meas}(\mathrm{RIEM(M)}, \pi_1(M)) \\ &\quad = \sum_{\lambda}^{\infty} \int\limits_{\mathrm{Hom}(\pi_1(M), G)/G} B_{\mathrm{Cov}}(\Delta^{\mathfrak{g}}, \lambda) \exp[-a\lambda]\, \mathrm{d}\nu([\theta]). \end{aligned} \tag{107}$$

This volume (107) of the corresponding space of riemannian structures depend on the bare cosmological constant, here in the role of a chemical potential controlling the average number of geodesic balls. It is related to the partition function, in the $\lambda \to \infty$ limit, of a discrete model of quantum gravity based on geodesic ball coverings (at least when the action can be reduced to the cosmological term). All this is strongly reminiscent of the interplay between two-dimensional quantum Yang–Mills theory and the intersection parings on moduli spaces of flat connections on a two-dimensional surface [Wt2]. In this connection it is worth stressing that the representation variety $\mathrm{Hom}(\pi_1(M), G)/G$ has a more direct geometrical meaning that evidentiates better the connection with quantum gravity rather than the usual interpretation as a moduli space for flat connection. In dimension two is known that, by taking $G \simeq PSL(2, \mathbb{R})$, the representation variety has a connected component homeomorphic to the Teichmüller space of the surface. Analogous Teichmüller components can be characterized for other choices of the group G (see e.g., [Go,Hi]) and thus by considering the representation variety in place of the moduli space of complex structure as is the case for 2D-gravity, implies that we are considering an extension of 2D-gravity. In dimension larger than two, the representation variety $\mathrm{Hom}(\pi_1(M), G)/G$ can be interpreted as the deformation space of local G-structures on M [Go]. For instance, if $G = \mathrm{O}(n)$ is the orthogonal group, then $\mathrm{Hom}(\pi_1(M), \mathrm{O}(n))/\mathrm{O}(n)$ is the moduli space of locally flat Euclidean structures on M.

This last remark thus explains why it is natural to sum over $\mathrm{Hom}(\pi_1(M), G)/G$. Indeed, since counting coverings can be thought of as an approximation to compute integrals over the space of riemannian structure, the sum over the representation variety $\mathrm{Hom}(\pi_1(M), G)/G$ is needed in order to take into account the *size* of the set of metrics realizing such G-structures in the space of all riemannian structure (of bounded geometry).

5.1. *Bounds on the critical exponents*

The relation between the counting function $B_{\mathrm{Cov}}(\Delta^{\mathfrak{g}}; \lambda)$ and the measures on the representations variety $\mathrm{Hom}(\pi_1(M), G)/G$ allows us to provide bounds on the critical exponents associated with $B_{\mathrm{Cov}}(\Delta^{\mathfrak{g}}; \lambda)$ by (formal) saddle-point estimation. A sounder application of this technique would require a deeper discussion of the properties of the measure $\nu([\theta])$ on $\mathrm{Hom}(\pi_1(M), G)/G$, in particular one needs to understand in detail the extension of a measure from the set of irreducible representations to the reducible ones corresponding to the singular points of the representation variety. We are not able to address this interesting question here. Nevertheless, we venture since the results obtained may be helpful. Let us fix our attention on the two-dimensional case first.

To begin with, let us be more specific on the choice of the group G into which we are considering representations of $\pi_1(M)$. A natural example is provided by $G = U(1)$. In such a case, the $U(1)$ conjugation action on $\mathrm{Hom}(\pi_1(M), U(1))/U(1)$ is trivial, and $\mathrm{Hom}(\pi_1(M), U(1))$ is just the Jacobian variety of the riemannian surface generated by the covering considered. Moreover, regardless of the complex structure, one has that topologically, $\mathrm{Hom}(\pi_1(M), U(1)) \simeq U(1)^{2h}$ where h is the genus of the surface (see e.g., [Go]). We can consider the average of (99), for $n = 2$, as the representation θ runs over $\mathrm{Hom}(\pi_1(M), U(1))$. Namely

$$\frac{2}{\sqrt{6\pi}}\left[\frac{4^4}{3^3}\right]^{\lambda}\int\limits_{\mathrm{Hom}(\pi_1(M),\,U(1))}\frac{(\tilde{w})^{\lambda}}{\Delta^{\mathfrak{q}}(M)}\lambda^{-1/2}\left(1+\mathrm{O}(\lambda^{-3/2})\right)\mathrm{d}\nu([\theta]). \tag{108}$$

On applying Laplace method, and denoting by Hom_0 the finite set in $\mathrm{Hom}(\pi_1(M), U(1))$ where the differential of $\log\tilde{w}$ vanishes and where the corresponding Hessian is a non-degenerate quadratic form, we can estimate the above integral in terms of $\lambda^{1/2}$ (which is the power of λ characterizing the subleading asymptotics in (99)) and obtain the bound

$$\int\limits_{\mathrm{Hom}(\pi_1(M),\,G)/G} B_{\mathrm{Cov}}(\Delta^{\mathfrak{q}},\lambda)$$

$$\leq \frac{2(2\pi)^h}{\sqrt{6\pi}}\left[\frac{4^4}{3^3}\right]^{\lambda}\sum_{\theta\in\mathrm{Hom}_0}\sqrt{a_\theta}\,\frac{(\tilde{w}_\theta)^{\lambda}}{\Delta^{\mathfrak{q}}_{\theta}(M)}\lambda^{-h/2-1/2}\left(1+\mathrm{O}(\lambda^{-3/2})\right), \tag{109}$$

where a_θ is the inverse of the determinant of the Hessian of $\log\tilde{w}$.

As recalled in the introductory remarks, we define the critical exponent $\eta(G)$ associated with the entropy function $B_{\mathrm{Cov}}(\Delta^{\mathfrak{q}},\lambda)$ by means of the relation

$$\int\limits_{\mathrm{Hom}(\pi_1(M),\,G)/G} B_{\mathrm{Cov}}(\Delta^{\mathfrak{q}},\lambda) \equiv \mathrm{Meas}\left(\frac{\mathrm{Hom}(\pi_1(M),G)}{G}\right)\exp[c\lambda]\lambda^{\eta_{\mathrm{sup}}-3}, \tag{110}$$

where c is a suitable constant (depending on G). Thus, corresponding to (109) we get the following upper bound for the critical exponent $\eta(G)$,

$$\eta(G=U(1)) \leq 2+\tfrac{1}{2}(1-h). \tag{111}$$

One may wish to compare this bound with the exact critical exponent associated with (1), namely

$$\eta_{\mathrm{Sup}} = 2+(1-h)\left(\frac{c-25-\sqrt{(25-c)(1-c)}}{12}\right). \tag{112}$$

It follows that (111) correctly reproduces the KPZ scaling in the case $h=1$ (notice however that $\eta=2$ is not a good testing ground since this value of the critical exponent holds for genus $h=1$ surfaces regardless both of the presence of matter and of the fluctuations of the metric geometry [D1]). The bound (111) is strict both for genus $h=0$ and $h>1$, and it remains consistent with KPZ scaling. One may suspect that it may be also consistent with a strong coupling of 2D-gravity with matter, namely in the regime where KPZ is believed not to be reliable. As a matter of fact, conformal field theory has not been used in deriving our entropy estimates. To discuss this point further, let us extend the above analysis to representations in more general groups.

Recall that the group G is endowed with an Ad-invariant, symmetric, non-degenerate bilinear form. This metric induces [Go], for $n = 2$, a symplectic structure on $\mathrm{Hom}(\pi_1(M),G)/G$, which can be used to give meaning to the integration of (99) over $\mathrm{Hom}(\pi_1(M),G)/G$, similarly to what was done in (108).

More in detail, if we denote by $z(\theta)$ the centralizer of $\theta(\pi_1(M))$ in G, then the dimension of the Zariski tangent space $H^1_{\mathfrak{g}}(M)$ to $\mathrm{Hom}(\pi_1(M), G)/G$ at θ is given by [Go,Wa]

$$(2h-2)\dim(G) + 2\dim(z(\theta)). \tag{113}$$

Thus, again on formal application of Laplace method, we get the bound (up to the usual exponential factor $(\frac{4^4}{3^3})^{\lambda}$),

$$\sum_{\theta\in\mathrm{Hom}_0} (2\pi)^{(h-1)\dim(G)+\dim(z(\theta))} \times\sqrt{a_\theta}\frac{\tilde{w}^{\lambda}_{\theta}}{\Delta^{\mathfrak{g}}_{\theta}(\mathcal{O}_h)}\lambda^{[-(h-1)\dim(G)/2-\dim(z(\theta))/2-1/2]}(1+\cdots) \tag{114}$$

with obvious meaning of Hom_0, and where $\cdots$ stands for terms of the order

$$\mathrm{O}(\lambda^{[-(h-1)\dim(G)/2-\dim(z(\theta))/2-3/2]}).$$

The corresponding bound to the critical exponent is (for a given $\theta\in\mathrm{Hom}_0$),

$$\eta(G)\leq 2+(1-h)\tfrac{1}{2}\dim(G)+\tfrac{1}{2}(1-\dim(z(\theta))). \tag{115}$$

As for the $G = U(1)$ case, the structure of this critical exponent is consistent with KPZ scaling, and it may be a good starting point for discussing a strong coupling regime between matter and 2D-gravity.

5.2. *The four-dimensional case*

The four-dimensional case can be readily discussed along the same lines of the two-dimensional case.

By (formally) integrating (99) over $\mathrm{Hom}(\pi_1(M), G)/G$, and again on applying Laplace method, we get the asymptotics

$$\sqrt{\frac{6}{5}}\left[\frac{6^6}{5^5}\right]^{\lambda}\sum_{\theta\in\mathrm{Hom}_0}(2\pi)^{-\dim(G)\chi(M)/4+b(2)/4-1/2} \times\sqrt{a_\theta}\frac{\tilde{w}^{\lambda}_{\theta}}{\Delta^{\mathfrak{g}}_{\theta}(\mathcal{O}_h)}\lambda^{[\dim(G)\chi(M)/8-b(2)/8-1/2]}(1+\cdots), \tag{116}$$

where $\cdots$ stand for terms of the order $\mathrm{O}(\lambda^{[\dim(G)\chi(M)/8-b(2)/8-3/2]})$. As usual, Hom_0 denotes the finite set in $\mathrm{Hom}(\pi_1(M), G)$ where the differential of $\log\tilde{w}$ vanishes and where the corresponding Hessian is a non-degenerate quadratic form. Notice that in the above asymptotics we used (42) providing the formal dimension of the Zariski tangent space to $\mathrm{Hom}(\pi_1(M), G)/G$. Notice also that in the above expression we can set $\Delta^{\mathfrak{g}}_{\theta}(\mathcal{O}_h) = 1$ (the torsion being trivial in dimension four for a closed manifold–see the remarks in Section 3.2; the same holds in dimension two).

The bound to the critical exponent corresponding to the estimate (116) is (for a given $\theta \in \mathrm{Hom}_0$),

$$\eta(G) \leq \tfrac{5}{2} + \tfrac{1}{8}\dim(G)\chi(M) - \tfrac{1}{8}b(2). \tag{117}$$

As recalled in the introductory remarks this bound is fully consistent with the (limited) numerical evidence at our disposal. In our opinion, a more careful treatment of the integration over the representation variety may considerably improve (also in the two-dimensional case) these bounds. We will not address these interesting questions any further here. In particular, one needs to understand in considerable detail the geometry of $\mathrm{Hom}(\pi_1(M), G)/G$, for $n \geq 3$. For instance, the rather naive approach to integration over the representation variety adopted above is not suitable for the three-dimensional case. In dimension 3 the Reidemeister torsion is not trivial, and integration over $\mathrm{Hom}(\pi_1(M), G)/G$ is rather delicate (see e.g., [JW] for a remarkable analysis) and a separate study is needed for discussing the three-dimensional case in full detail. A look at (99) shows that the entropy estimates, for $n = 3$, have exactly the structure one would expect in this case. Indeed the integration of the (Ray–Singer) torsion over a moduli space of flat connections (our $\mathrm{Hom}(\pi_1(M), G)/G$), is the basic ingredient in Witten's approach to 3D-gravity [Sc]. Details on this case will be presented in a forthcoming paper.

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Note added in proof

A simple proof of the entropy bound, and an actual entropy estimate for Dynamically Triangulated Manifolds in dimensions 3 and 4, is now available: M. Carfora and A. Marzuoli, Holonomy and entropy estimates for dynamically triangulated manifolds, in: Quantum Geometry and Diff-invariant Quantum Field Theory, special issue of J. Math. Phys., to appear in Nov. 95.

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